MATHEMATICS

(For 11th Class)

ਇਹ ਪੁਸਤਕ ਪੰਜਾਬ ਸਰਕਾਰ ਦੁਆਰਾ ਮੁਫਤ
ਦਿੱਤੀ ਜਾਣੀ ਹੈ ਅਤੇ ਵਿਕਰੀ ਲਈ ਨਹੀਂ ਹੈ।

PUNJAB SCHOOL EDUCATION BOARD

Sahibzada Ajit Singh Nagar

Edition 2023 .... 16,000 Copies

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ਇਹ ਪੁਸਤਕ ਵਿਕਰੀ ਲਈ ਨਹੀਂ ਹੈ।

Published by : Secretary, Punjab School Education Board, Vidya Bhavan, Phase-8, Sahibzada Ajit Singh Nagar-160062 & Printed by **Nav Durga Offset Printers, Meerut**

Foreword

Punjab School Education Board, has been continuously engaged in preparation and review of syllabi and textbooks. In today's scenario, imparting right education to students is the joint responsibility of teachers as well as parents. With a view to carry out entrusted responsibility, some important changes pertaining to present day educational requirements have been made in textbooks and syllabus in accordance with NCF 2005.

Mathematics has an important place in school curriculum and a good textbook is the first requisite to achieve desired learning outcomes. Therefore, the content matter of Mathematics-XI has been so arranged so as to develop reasoning power of the students and to enhance their understanding of the subject. Graded questions and exercises have been given to suit the mental level of the students. This book is prepared by NCERT, New Delhi for class XI and is being published by Punjab School Education Board, with the permission of NCERT, New Delhi.

Every effort has been made to make the book useful for students as well as for the teachers. However, constructive suggestions for its further improvement would be gratefully acknowledged.

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Contents

$$P(A \cup B) = P(A) + P(B - A) \quad \ldots (2)$$

and $$P(B) = P(A \cap B) + P(B - A) \quad \ldots (3)$$

Subtracting (3) from (2) gives

$$P(A \cup B) - P(B) = P(A) - P(A \cap B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The above result can further be verified by observing the Venn Diagram (Fig 14.1)

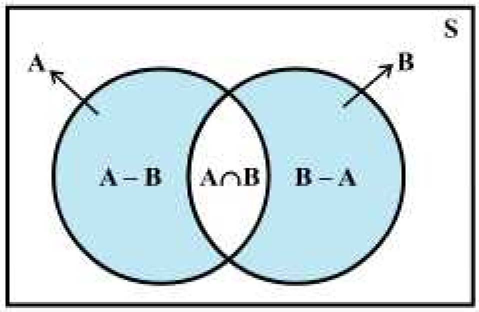

Fig 14.1

If A and B are disjoint sets, i.e., they are mutually exclusive events, then $A \cap B = \phi$

Therefore $P(A \cap B) = P(\phi) = 0$

Thus, for mutually exclusive events A and B, we have

$$P(A \cup B) = P(A) + P(B),$$

which is Axiom (iii) of probability.

14.2.4 ***Probability of event 'not A'*** Consider the event A = {2, 4, 6, 8} associated with the experiment of drawing a card from a deck of ten cards numbered from 1 to 10. Clearly the sample space is S = {1, 2, 3, ...,10}

If all the outcomes 1, 2, ...,10 are considered to be equally likely, then the probability of each outcome is $\frac{1}{10}$

Now $$P(A) = P(2) + P(4) + P(6) + P(8)$$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

Also event 'not A' = A′ = {1, 3, 5, 7, 9, 10}

Now $$P(A') = P(1) + P(3) + P(5) + P(7) + P(9) + P(10)$$

$$= \frac{6}{10} = \frac{3}{5}$$

Thus, $P(A') = \frac{3}{5} = 1 - \frac{2}{5} = 1 - P(A)$

Also, we know that A′ and A are mutually exclusive and exhaustive events i.e.,

$A \cap A' = \phi$ and $A \cup A' = S$

or $P(A \cup A') = P(S)$

Now $P(A) + P(A') = 1$, by using axioms (ii) and (iii).

or $P(A') = P(\text{not } A) = 1 - P(A)$

We now consider some examples and exercises having equally likely outcomes unless stated otherwise.

Example 5 One card is drawn from a well shuffled deck of 52 cards. If each outcome is equally likely, calculate the probability that the card will be

(i) a diamond (ii) not an ace
(iii) a black card (i.e., a club or, a spade) (iv) not a diamond
(v) not a black card.

Solution When a card is drawn from a well shuffled deck of 52 cards, the number of possible outcomes is 52.

(i) Let A be the event 'the card drawn is a diamond'
Clearly the number of elements in set A is 13.

Therefore, $P(A) = \frac{13}{52} = \frac{1}{4}$

i.e. probability of a diamond card $= \frac{1}{4}$

(ii) We assume that the event 'Card drawn is an ace' is B

Therefore 'Card drawn is not an ace' should be B′.

We know that $P(B') = 1 - P(B) = 1 - \frac{4}{52} = 1 - \frac{1}{13} = \frac{12}{13}$

(iii) Let C denote the event 'card drawn is black card'

Therefore, number of elements in the set C = 26

i.e. $P(C) = \frac{26}{52} = \frac{1}{2}$

Thus, probability of a black card = $\frac{1}{2}$.

(iv) We assumed in (i) above that A is the event 'card drawn is a diamond', so the event 'card drawn is not a diamond' may be denoted as A' or 'not A'

Now P(not A) = 1 – P(A) = $1-\frac{1}{4}=\frac{3}{4}$

(v) The event 'card drawn is not a black card' may be denoted as C′ or 'not C'.

We know that P(not C) = 1 – P(C) = $1-\frac{1}{2}=\frac{1}{2}$

Therefore, probability of not a black card = $\frac{1}{2}$

Example 6 A bag contains 9 discs of which 4 are red, 3 are blue and 2 are yellow. The discs are similar in shape and size. A disc is drawn at random from the bag. Calculate the probability that it will be (i) red, (ii) yellow, (iii) blue, (iv) not blue, (v) either red or blue.

Solution There are 9 discs in all so the total number of possible outcomes is 9.
Let the events A, B, C be defined as

A: 'the disc drawn is red'

B: 'the disc drawn is yellow'

C: 'the disc drawn is blue'.

(i) The number of red discs = 4, i.e., n (A) = 4

Hence $\quad$ P(A) = $\frac{4}{9}$

(ii) The number of yellow discs = 2, i.e., n (B) = 2

Therefore, $\quad$ P(B) = $\frac{2}{9}$

(iii) The number of blue discs = 3, i.e., n(C) = 3

Therefore, $\quad$ P(C) = $\frac{3}{9}=\frac{1}{3}$

(iv) Clearly the event 'not blue' is 'not C'. We know that P(not C) = 1 – P(C)

Therefore $\quad P(\text{not } C) = 1-\frac{1}{3}=\frac{2}{3}$

(v) The event 'either red or blue' may be described by the set 'A or C'

Since, A and C are mutually exclusive events, we have

$$P(\text{A or C}) = P(A \cup C) = P(A) + P(C) = \frac{4}{9}+\frac{1}{3}=\frac{7}{9}$$

Example 7 Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination.

(b) Atleast one of them will not qualify the examination and

(c) Only one of them will qualify the examination.

Solution Let E and F denote the events that Anil and Ashima will qualify the examination, respectively. Given that

$$P(E) = 0.05, P(F) = 0.10 \text{ and } P(E \cap F) = 0.02.$$

Then

(a) The event 'both Anil and Ashima will not qualify the examination' may be expressed as $E' \cap F'$.

Since, E' is 'not E', i.e., Anil will not qualify the examination and F' is 'not F', i.e., Ashima will not qualify the examination.

Also $\quad E' \cap F' = (E \cup F)'$ (by Demorgan's Law)

Now $\quad P(E \cup F) = P(E) + P(F) - P(E \cap F)$

or $\quad P(E \cup F) = 0.05 + 0.10 - 0.02 = 0.13$

Therefore $P(E' \cap F') = P(E \cup F)' = 1 - P(E \cup F) = 1 - 0.13 = 0.87$

(b) P (atleast one of them will not qualify)

= 1 − P(both of them will qualify)

= 1 − 0.02 = 0.98

(c) The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima will qualify) i.e., $E \cap F'$ or $E' \cap F$, where $E \cap F'$ and $E' \cap F$ are mutually exclusive.

Therefore, P(only one of them will qualify) $= P(E \cap F' \text{ or } E' \cap F)$

$$= P(E \cap F') + P(E' \cap F) = P(E) - P(E \cap F) + P(F) - P(E \cap F)$$

$$= 0.05 - 0.02 + 0.10 - 0.02 = 0.11$$

Example 8 A committee of two persons is selected from two men and two women. What is the probability that the committee will have (a) no man? (b) one man? (c) two men?

Solution The total number of persons = 2 + 2 = 4. Out of these four person, two can be selected in ${}^4C_2$ ways.

(a) No men in the committee of two means there will be two women in the committee. Out of two women, two can be selected in ${}^2C_2 = 1$ way.

Therefore $$P(\text{no man}) = \frac{{}^2C_2}{{}^4C_2} = \frac{1 \times 2 \times 1}{4 \times 3} = \frac{1}{6}$$

(b) One man in the committee means that there is one woman. One man out of 2 can be selected in ${}^2C_1$ ways and one woman out of 2 can be selected in ${}^2C_1$ ways. Together they can be selected in ${}^2C_1 \times {}^2C_1$ ways.

Therefore $$P(\text{One man}) = \frac{{}^2C_1 \times {}^2C_1}{{}^4C_2} = \frac{2 \times 2}{2 \times 3} = \frac{2}{3}$$

(c) Two men can be selected in ${}^2C_2$ way.

Hence $$P(\text{Two men}) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{{}^4C_2} = \frac{1}{6}$$

EXERCISE 14.2

1. Which of the following can not be valid assignment of probabilities for outcomes of sample Space $S = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7\}$

Assignment	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
(a)	0.1	0.01	0.05	0.03	0.01	0.2	0.6
(b)	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
(c)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
(d)	– 0.1	0.2	0.3	0.4	– 0.2	0.1	0.3
(e)	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{15}{14}$

2. A coin is tossed twice, what is the probability that atleast one tail occurs?
3. A die is thrown, find the probability of following events:
   (i) A prime number will appear,
   (ii) A number greater than or equal to 3 will appear,
   (iii) A number less than or equal to one will appear,
   (iv) A number more than 6 will appear,
   (v) A number less than 6 will appear.
4. A card is selected from a pack of 52 cards.
   (a) How many points are there in the sample space?
   (b) Calculate the probability that the card is an ace of spades.
   (c) Calculate the probability that the card is (i) an ace (ii) black card.
5. A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. find the probability that the sum of numbers that turn up is (i) 3 (ii) 12
6. There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?
7. A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up.
   From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.
8. Three coins are tossed once. Find the probability of getting

(i) 3 heads	(ii) 2 heads	(iii) atleast 2 heads
(iv) atmost 2 heads	(v) no head	(vi) 3 tails
(vii) exactly two tails	(viii) no tail	(ix) atmost two tails

9. If $\frac{2}{11}$ is the probability of an event, what is the probability of the event 'not A'.
10. A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is (i) a vowel (ii) a consonant

11. In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game? [**Hint** order of the numbers is not important.]

12. Check whether the following probabilities P(A) and P(B) are consistently defined
(i) $P(A) = 0.5,\ P(B) = 0.7,\ P(A \cap B) = 0.6$
(ii) $P(A) = 0.5, P(B) = 0.4, P(A \cup B) = 0.8$

13. Fill in the blanks in following table:

	P(A)	P(B)	P(A ∩ B)	P(A ∪ B)
(i)	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{15}$	. . .
(ii)	0.35	. . .	0.25	0.6
(iii)	0.5	0.35	. . .	0.7

14. Given $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find P(A or B), if A and B are mutually exclusive events.

15. If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and P(E and F) = $\frac{1}{8}$, find (i) P(E or F), (ii) P(not E and not F).

16. Events E and F are such that P(not E or not F) = 0.25, State whether E and F are mutually exclusive.

17. A and B are events such that P(A) = 0.42, P(B) = 0.48 and P(A and B) = 0.16. Determine (i) P(not A), (ii) P(not B) and (iii) P(A or B)

18. In Class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology.

19. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

20. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, what is the probability of passing the Hindi examination?

21. In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(i) The student opted for NCC or NSS.

(ii) The student has opted neither NCC nor NSS.

(iii) The student has opted NSS but not NCC.

Miscellaneous Examples

Example 9 On her vacations Veena visits four cities (A, B, C and D) in a random order. What is the probability that she visits

(i) A before B? (ii) A before B and B before C?

(iii) A first and B last? (iv) A either first or second?

(v) A just before B?

Solution The number of arrangements (orders) in which Veena can visit four cities A, B, C, or D is 4! i.e., 24.Therefore, n (S) = 24.

Since the number of elements in the sample space of the experiment is 24 all of these outcomes are considered to be equally likely. A sample space for the experiment is

S = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB
BACD, BADC, BDAC, BDCA, BCAD, BCDA
CABD, CADB, CBDA, CBAD, CDAB, CDBA
DABC, DACB, DBCA, DBAC, DCAB, DCBA}

(i) Let the event 'she visits A before B' be denoted by E

Therefore, E = {ABCD, CABD, DABC, ABDC, CADB, DACB
ACBD, ACDB, ADBC, CDAB, DCAB, ADCB}

Thus $$\mathrm{P}(\mathrm{E})=\frac{n(\mathrm{E})}{n(\mathrm{S})}=\frac{12}{24}=\frac{1}{2}$$

(ii) Let the event 'Veena visits A before B and B before C' be denoted by F.

Here F = {ABCD, DABC, ABDC, ADBC}

Therefore, $$\mathrm{P}(\mathrm{F})=\frac{n(\mathrm{F})}{n(\mathrm{S})}=\frac{4}{24}=\frac{1}{6}$$

Students are advised to find the probability in case of (iii), (iv) and (v).

Example 10 Find the probability that when a hand of 7 cards is drawn from a well shuffled deck of 52 cards, it contains (i) all Kings (ii) 3 Kings (iii) atleast 3 Kings.

Solution Total number of possible hands = ${}^{52}C_7$

(i) Number of hands with 4 Kings = ${}^{4}C_4 \times {}^{48}C_3$ (other 3 cards must be chosen from the rest 48 cards)

Hence P (a hand will have 4 Kings) = $\dfrac{{}^{4}C_4 \times {}^{48}C_3}{{}^{52}C_7} = \dfrac{1}{7735}$

(ii) Number of hands with 3 Kings and 4 non-King cards = ${}^{4}C_3 \times {}^{48}C_4$

Therefore P (3 Kings) = $\dfrac{{}^{4}C_3 \times {}^{48}C_4}{{}^{52}C_7} = \dfrac{9}{1547}$

(iii) P(atleast 3 King) = P(3 Kings or 4 Kings)
= P(3 Kings) + P(4 Kings)

$$= \frac{9}{1547} + \frac{1}{7735} = \frac{46}{7735}$$

Example 11 If A, B, C are three events associated with a random experiment, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Solution Consider E = B ∪ C so that

$$P(A \cup B \cup C) = P(A \cup E)$$

$$= P(A) + P(E) - P(A \cap E) \qquad \text{... (1)}$$

Now

$$P(E) = P(B \cup C)$$

$$= P(B) + P(C) - P(B \cap C) \qquad \text{... (2)}$$

Also $A \cap E = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [using distribution property of intersection of sets over the union]. Thus

$$P(A \cap E) = P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[A \cap B \cap C] \quad \text{... (3)}$$

Using (2) and (3) in (1), we get

$$P[A \cup B \cup C] = P(A) + P(B) + P(C) - P(B \cap C)$$
$$- P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Example 12 In a relay race there are five teams A, B, C, D and E.

(a) What is the probability that A, B and C finish first, second and third, respectively.

(b) What is the probability that A, B and C are first three to finish (in any order) (Assume that all finishing orders are equally likely)

Solution If we consider the sample space consisting of all finishing orders in the first three places, we will have $^5P_3$, i.e., $\frac{5!}{(5-3)!} = 5 \times 4 \times 3 = 60$ sample points, each with a probability of $\frac{1}{60}$.

(a) A, B and C finish first, second and third, respectively. There is only one finishing order for this, i.e., ABC.

Thus P(A, B and C finish first, second and third respectively) = $\frac{1}{60}$.

(b) A, B and C are the first three finishers. There will be 3! arrangements for A, B and C. Therefore, the sample points corresponding to this event will be 3! in number.

So P (A, B and C are first three to finish) $= \frac{3!}{60} = \frac{6}{60} = \frac{1}{10}$

Miscellaneous Exercise on Chapter 14

1. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that
(i) all will be blue? (ii) atleast one will be green?
2. 4 cards are drawn from a well – shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade?

3. A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(i) P(2) (ii) P(1 or 3) (iii) P(not 3)

4. In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets.

5. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?

(b) you both enter the different sections?

6. Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter is in its proper envelope.

7. A and B are two events such that P(A) = 0.54, P(B) = 0.69 and $P(A \cap B) = 0.35$. Find (i) $P(A \cup B)$ (ii) $P(A' \cap B')$ (iii) $P(A \cap B')$ (iv) $P(B \cap A')$

8. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. Particulars of five persons are as follows:

S. No.	Name	Sex	Age in years
1.	Harish	M	30
2.	Rohan	M	33
3.	Sheetal	F	46
4.	Alis	F	28
5.	Salim	M	41

A person is selected at random from this group to act as a spokesperson. What is the probability that the spokesperson will be either male or over 35 years?

9. If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when, (i) the digits are repeated? (ii) the repetition of digits is not allowed?

10. The number lock of a suitcase has 4 wheels, each labelled with ten digits i.e., from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase?

Summary

In this Chapter, we studied about the axiomatic approach of probability. The main features of this Chapter are as follows:

- ***Event***: A subset of the sample space
- ***Impossible event*** : The empty set
- ***Sure event***: The whole sample space
- ***Complementary event or 'not event'*** : The set A′ or S – A
- ***Event* A *or* B**: The set A ∪ B
- ***Event* A *and* B**: The set A ∩ B
- ***Event* A *and* not B**: The set A – B
- ***Mutually exclusive event***: A and B are mutually exclusive if $A \cap B = \phi$
- ***Exhaustive and mutually exclusive events***: Events $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive if $E_1 \cup E_2 \cup ... \cup E_n = S$ and $E_i \cap E_j = \phi \ \forall\, i \neq j$
- ***Probability***: Number P (ω_i) associated with sample point ω_i such that

  (i) $0 \leq P(\omega_i) \leq 1$ (ii) $\sum P(\omega_i)$ for all $\omega_i \in S = 1$

  (iii) $P(A) = \sum P(\omega_i)$ for all $\omega_i \in A$. The number P (ω_i) is called *probability of the outcome* ω_i.
- ***Equally likely outcomes***: All outcomes with equal probability
- ***Probability of an event***: For a finite sample space with equally likely outcomes

  Probability of an event $P(A) = \dfrac{n(A)}{n(S)}$, where $n(A)$ = number of elements in the set A, $n(S)$ = number of elements in the set S.
- If A and B are any two events, then

  P(A or B) = P(A) + P(B) – P(A and B)
  equivalently, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are mutually exclusive, then P(A or B) = P(A) + P(B)
- If A is any event, then

  P(not A) = 1 – P(A)

Historical Note

Probability theory like many other branches of mathematics, evolved out of practical consideration. It had its origin in the 16th century when an Italian physician and mathematician Jerome Cardan (1501–1576) wrote the first book on the subject "Book on Games of Chance" (Biber de Ludo Aleae). It was published in 1663 after his death.

In 1654, a gambler Chevalier de Metre approached the well known French Philosopher and Mathematician Blaise Pascal (1623–1662) for certain dice problem. Pascal became interested in these problems and discussed with famous French Mathematician Pierre de Fermat (1601–1665). Both Pascal and Fermat solved the problem independently. Besides, Pascal and Fermat, outstanding contributions to probability theory were also made by Christian Huygenes (1629–1665), a Dutchman, J. Bernoulli (1654–1705), De Moivre (1667–1754), a Frenchman Pierre Laplace (1749–1827), the Russian P.L Chebyshev (1821–1897), A. A Markov (1856–1922) and A. N Kolmogorove (1903–1987). Kolmogorov is credited with the axiomatic theory of probability. His book 'Foundations of Probability' published in 1933, introduces probability as a set function and is considered a classic.

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Appendix 1

INFINITE SERIES

A.1.1 Introduction

As discussed in the Chapter 9 on Sequences and Series, a sequence $a_1, a_2, ..., a_n, ...$ having infinite number of terms is called *infinite sequence* and its indicated sum, i.e., $a_1 + a_2 + a_3 + ... + a_n + ...$ is called an *infinte series* associated with infinite sequence. This series can also be expressed in abbreviated form using the sigma notation, i.e.,

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k$$

In this Chapter, we shall study about some special types of series which may be required in different problem situations.

A.1.2 Binomial Theorem for any Index

In Chapter 8, we discussed the Binomial Theorem in which the index was a positive integer. In this Section, we state a more general form of the theorem in which the index is not necessarily a whole number. It gives us a particular type of infinite series, called *Binomial Series*. We illustrate few applications, by examples.

We know the formula

$$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + \ldots + {}^nC_n\, x^n$$

Here, n is non-negative integer. Observe that if we replace index n by negative integer or a fraction, then the combinations ${}^nC_r$ do not make any sense.

We now state (without proof), the Binomial Theorem, giving an infinite series in which the index is negative or a fraction and not a whole number.

Theorem The formula

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \ldots$$

holds whenever $|x| < 1$.

Remark 1. Note carefully the condition $|x| < 1$, i.e., $-1 < x < 1$ is necessary when m is negative integer or a fraction. For example, if we take $x = -2$ and $m = -2$, we obtain

$$(1-2)^{-2} = 1 + (-2)(-2) + \frac{(-2)(-3)}{1.2}(-2)^2 + ...$$

or $\quad 1 = 1 + 4 + 12 + \ldots$

This is not possible

2. Note that there are infinite number of terms in the expansion of $(1+x)^m$, when m is a negative integer or a fraction

Consider $\quad (a+b)^m = \left[a\left(1+\frac{b}{a}\right)\right]^m = a^m\left(1+\frac{b}{a}\right)^m$

$$= a^m\left[1 + m\frac{b}{a} + \frac{m(m-1)}{1.2}\left(\frac{b}{a}\right)^2 + ...\right]$$

$$= a^m + ma^{m-1}b + \frac{m(m-1)}{1.2}a^{m-2}b^2 + ...$$

This expansion is valid when $\left|\frac{b}{a}\right| < 1$ or equivalently when $|b| < |a|$.

The general term in the expansion of $(a+b)^m$ is

$$\frac{m(m-1)(m-2)...(m-r+1)a^{m-r}b^r}{1.2.3...r}$$

We give below certain particular cases of Binomial Theorem, when we assume $|x| < 1$, these are left to students as exercises:

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

Example 1 Expand $\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}$, when $|x| < 2$.

Solution We have

$$\left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = 1+\frac{\left(-\frac{1}{2}\right)}{1}\left(\frac{-x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2}\left(-\frac{x}{2}\right)^2+...$$

$$= 1+\frac{x}{4}+\frac{3x^2}{32}+...$$

A.1.3 Infinite Geometric Series

From Chapter 9, Section 9.5, a sequence a_1, a_2, a_3, ..., a_n is called G.P., if $\frac{a_{k+1}}{a_k} = r$ (constant) for k = 1, 2, 3, ..., n–1. Particularly, if we take $a_1 = a$, then the resulting sequence $a, ar, ar^2, ..., ar^{n-1}$ is taken as the standard form of G.P., where a is first term and r, the common ratio of G.P.

Earlier, we have discussed the formula to find the sum of finite series $a + ar + ar^2 + ... + ar^{n-1}$ which is given by

$$S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

In this section, we state the formula to find the sum of infinite geometric series $a + ar + ar^2 + ... + ar^{n-1} + ...$ and illustrate the same by examples.

Let us consider the G.P. 1, $\frac{2}{3}$, $\frac{4}{9}$,...

Here $a = 1$, $r = \frac{2}{3}$. We have

$$S_n = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right] \qquad ... (1)$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger.

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, $\left(\frac{2}{3}\right)^n$ becomes sufficiently small. In other words, as $n \to \infty, \left(\frac{2}{3}\right)^n \to 0$. Consequently, we find that the sum of infinitely many terms is given by S = 3.

Thus, for infinite geometric progression $a, ar, ar^2, ...$, if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case, $r^n \to 0$ as $n \to \infty$ since $|r|<1$ and then $\frac{ar^n}{1-r} \to 0$. Therefore,

$$S_n \to \frac{a}{1-r} \text{ as } n \to \infty.$$

Symbolically, sum to infinity of infinite geometric series is denoted by S. Thus, we have $\mathrm{S} = \frac{a}{1-r}$

For example

(i) $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = \frac{1}{1-\frac{1}{2}} = 2$

(ii) $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + ... = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

Example 2 Find the sum to infinity of the G.P. ;

$$\frac{-5}{4}, \frac{5}{16}, \frac{-5}{64}, \ldots.$$

Solution Here $a = \frac{-5}{4}$ and $r = -\frac{1}{4}$. Also $|r| < 1$.

Hence, the sum to infinity is $\dfrac{\frac{-5}{4}}{1+\frac{1}{4}} = \dfrac{\frac{-5}{4}}{\frac{5}{4}} = -1$.

A.1.4 Exponential Series

Leonhard Euler (1707 – 1783), the great Swiss mathematician introduced the number e in his calculus text in 1748. The number e is useful in calculus as π in the study of the circle.

Consider the following infinite series of numbers

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \quad \ldots (1)$$

The sum of the series given in (1) is denoted by the number e

Let us estimate the value of the number e.

Since every term of the series (1) is positive, it is clear that its sum is also positive.

Consider the two sums

$$\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots + \frac{1}{n!} + \ldots \quad \ldots (2)$$

and

$$\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots. + \frac{1}{2^{n-1}} + \ldots \quad \ldots (3)$$

Observe that

$$\frac{1}{3!} = \frac{1}{6} \text{ and } \frac{1}{2^2} = \frac{1}{4}, \text{ which gives } \frac{1}{3!} < \frac{1}{2^2}$$

$$\frac{1}{4!} = \frac{1}{24} \text{ and } \frac{1}{2^3} = \frac{1}{8}, \text{ which gives } \frac{1}{4!} < \frac{1}{2^3}$$

$$\frac{1}{5!} = \frac{1}{120} \text{ and } \frac{1}{2^4} = \frac{1}{16}, \text{ which gives } \frac{1}{5!} < \frac{1}{2^4}.$$

Therefore, by analogy, we can say that

$$\frac{1}{n!} < \frac{1}{2^{n-1}}, \text{ when } n > 2$$

We observe that each term in (2) is less than the corresponding term in (3),

Therefore $$\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...+\frac{1}{n!}\right)<\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right) \quad ... (4)$$

Adding $\left(1+\frac{1}{1!}+\frac{1}{2!}\right)$ on both sides of (4), we get,

$$\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...+\frac{1}{n!}+...\right)$$

$$<\left\{\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right)\right\} \quad ... (5)$$

$$=\left\{1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right)\right\}$$

$$=1+\frac{1}{1-\frac{1}{2}}=1+2 \;=3$$

Left hand side of (5) represents the series (1). Therefore $e < 3$ and also $e > 2$ and hence $2 < e < 3$.

Remark The exponential series involving variable x can be expressed as

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

Example 3 Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of x.

Solution In the exponential series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...$$

replacing x by $(2x + 3)$, we get

$$e^{2x+3} = 1+\frac{(2x+3)}{1!}+\frac{(2x+3)^2}{2!}+...$$

Here, the general term is $\frac{(2x+3)^n}{n!} = \frac{(3+2x)^n}{n!}$. This can be expanded by the Binomial Theorem as

$$\frac{1}{n!}\left[3^n + {}^nC_1 3^{n-1}(2x) + {}^nC_2 3^{n-2}(2x)^2 + ... + (2x)^n\right].$$

Here, the coefficient of x^2 is $\frac{{}^nC_2 3^{n-2} 2^2}{n!}$. Therefore, the coefficient of x^2 in the whole series is

$$\sum_{n=2}^{\infty}\frac{{}^nC_2 3^{n-2}2^2}{n!} = 2\sum_{n=2}^{\infty}\frac{n(n-1)3^{n-2}}{n!}$$

$$= 2\sum_{n=2}^{\infty}\frac{3^{n-2}}{(n-2)!} \quad \text{[using } n! = n\,(n-1)\,(n-2)!\text{]}$$

$$= 2\left[1+\frac{3}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+...\right]$$

$$= 2e^3 .$$

Thus $2e^3$ is the coefficient of x^2 in the expansion of e^{2x+3}.

Alternatively $e^{2x+3} = e^3 \,.\, e^{2x}$

$$= e^3\left[1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+...\right]$$

Thus, the coefficient of x^2 in the expansion of e^{2x+3} is $e^3.\frac{2^2}{2!}=2e^3$

Example 4 Find the value of e^2, rounded off to one decimal place.

Solution Using the formula of exponential series involving x, we have

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

Putting $x = 2$, we get

$$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + ...$$

$$= 1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{4}{15} + \frac{4}{45} + ...$$

$$\geq \text{ the sum of first seven terms } \geq 7.355.$$

On the other hand, we have

$$e^2 < \left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right) + \frac{2^5}{5!}\left(1 + \frac{2}{6} + \frac{2^2}{6^2} + \frac{2^3}{6^3} + ...\right)$$

$$= 7 + \frac{4}{15}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + ...\right) = 7 + \frac{4}{15}\left(\frac{1}{1 - \frac{1}{3}}\right) = 7 + \frac{2}{5} = 7.4.$$

Thus, e^2 lies between 7.355 and 7.4. Therefore, the value of e^2, rounded off to one decimal place, is 7.4.

A.1.5 Logarithmic Series

Another very important series is logarithmic series which is also in the form of infinite series. We state the following result without proof and illustrate its application with an example.

Theorem If $|x| < 1$, then

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$$

The series on the right hand side of the above is called the *logarithmic series*.

Note The expansion of $\log_e (1+x)$ is valid for $x = 1$. Substituting $x = 1$ in the expansion of $\log_e (1+x)$, we get

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

Example 5 If α, β are the roots of the equation $x^2 - px + q = 0$, prove that

$$\log_e\left(1+px+qx^2\right)=(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2+\frac{\alpha^3+\beta^3}{3}x^3-...$$

Solution Right hand side $= \left[\alpha x-\frac{\alpha^2x^2}{2}+\frac{\alpha^3x^3}{3}-...\right]+\left[\beta x-\frac{\beta^2x^2}{2}+\frac{\beta^3x^3}{3}-...\right]$

$$= \log_e(1+\alpha x)+\log(1+\beta x)$$

$$= \log_e\left(1+(\alpha+\beta)x+\alpha\beta x^2\right)$$

$$= \log_e\left(1+px+qx^2\right) = \text{Left hand side.}$$

Here, we have used the facts $\alpha+\beta=p$ and $\alpha\beta=q$. We know this from the given roots of the quadratic equation. We have also assumed that both $|\alpha x|< 1$ and $|\beta x|< 1$.

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Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

Much of our progress in the last few centuries has made it necessary to apply mathematical methods to real-life problems arising from different fields – be it Science, Finance, Management etc. The use of Mathematics in solving real-world problems has become widespread especially due to the increasing computational power of digital computers and computing methods, both of which have facilitated the handling of lengthy and complicated problems. The process of translation of a real-life problem into a mathematical form can give a better representation and solution of certain problems. The process of translation is called Mathematical Modelling.

Here we shall familiaries you with the steps involved in this process through examples. We shall first talk about what a mathematical model is, then we discuss the steps involved in the process of modelling.

A.2.2 Preliminaries

Mathematical modelling is an essential tool for understanding the world. In olden days the Chinese, Egyptians, Indians, Babylonians and Greeks indulged in understanding and predicting the natural phenomena through their knowledge of mathematics. The architects, artisans and craftsmen based many of their works of art on geometric prinicples.

Suppose a surveyor wants to measure the height of a tower. It is physically very difficult to measure the height using the measuring tape. So, the other option is to find out the factors that are useful to find the height. From his knowledge of trigonometry, he knows that if he has an angle of elevation and the distance of the foot of the tower to the point where he is standing, then he can calculate the height of the tower.

So, his job is now simplified to find the angle of elevation to the top of the tower and the distance from the foot of the tower to the point where he is standing. Both of which are easily measurable. Thus, if he measures the angle of elevation as 40° and the distance as 450m, then the problem can be solved as given in Example 1.

Example 1 The angle of elevation of the top of a tower from a point O on the ground, which is 450 m away from the foot of the tower, is 40°. Find the height of the tower.

Solution We shall solve this in different steps.

Step 1 We first try to understand the real problem. In the problem a tower is given and its height is to be measured. Let h denote the height. It is given that the horizontal distance of the foot of the tower from a particular point O on the ground is 450 m. Let d denotes this distance. Then $d = 450$m. We also know that the angle of elevation, denoted by θ, is 40°.

The real problem is to find the height h of the tower using the known distance d and the angle of elevation θ.

Step 2 The three quantities mentioned in the problem are height, distance and angle of elevation.

So we look for a relation connecting these three quantities. This is obtained by expressing it geometrically in the following way (Fig 1).

AB denotes the tower. OA gives the horizontal distance from the point O to foot of the tower. $\angle$AOB is the angle of elevation. Then we have

$$\tan\theta = \frac{h}{d} \text{ or } h = d\tan\theta \qquad \ldots (1)$$

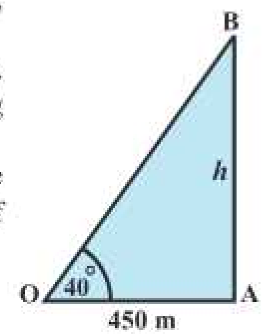


Fig 1

This is an equation connecting θ, h and d.

Step 3 We use Equation (1) to solve h. We have $\theta = 40°$, and $d = 450$m. Then we get $h = \tan 40° \times 450 = 450 \times 0.839 = 377.6$m

Step 4 Thus we got that the height of the tower approximately 378m.

Let us now look at the different steps used in solving the problem. In step 1, we have studied the real problem and found that the problem involves three parameters height, distance and angle of elevation. That means in this step we have *studied the real-life problem and identified the parameters.*

In the Step 2, we used some geometry and found that the problem can be represented geometrically as given in Fig 1. Then we used the trigonometric ratio for the "tangent" function and found the relation as

$$h = d\tan\theta$$

So, in this step we formulated the problem mathematically. That means we found an equation representing the real problem.

In Step 3, we solved the mathematical problem and got that $h = 377.6$m. That is we found

Solution of the problem.

In the last step, we interpreted the solution of the problem and stated that the height of the tower is approximately 378m. We call this as

Interpreting the mathematical solution to the real situation

In fact these are the steps mathematicians and others use to study various real-life situations. We shall consider the question, "why is it necessary to use mathematics to solve different situations."

Here are some of the examples where mathematics is used effectively to study various situations.

1. Proper flow of blood is essential to transmit oxygen and other nutrients to various parts of the body in humanbeings as well as in all other animals. Any constriction in the blood vessel or any change in the characteristics of blood vessels can change the flow and cause damages ranging from minor discomfort to sudden death. The problem is to find the relationship between blood flow and physiological characteristics of blood vessel.
2. In cricket a third umpire takes decision of a LBW by looking at the trajectory of a ball, simulated, assuming that the batsman is not there. Mathematical equations are arrived at, based on the known paths of balls before it hits the batsman's leg. This simulated model is used to take decision of LBW.
3. Meteorology department makes weather predictions based on mathematical models. Some of the parameters which affect change in weather conditions are temperature, air pressure, humidity, wind speed, etc. The instruments are used to measure these parameters which include thermometers to measure temperature, barometers to measure airpressure, hygrometers to measure humidity, anemometers to measure wind speed. Once data are received from many stations around the country and feed into computers for further analysis and interpretation.
4. Department of Agriculture wants to estimate the yield of rice in India from the standing crops. Scientists identify areas of rice cultivation and find the average yield per acre by cutting and weighing crops from some representative fields. Based on some statistical techniques decisions are made on the average yield of rice.

   How do mathematicians help in solving such problems? They sit with experts in the area, for example, a physiologist in the first problem and work out a mathematical equivalent of the problem. This equivalent consists of one or more equations or inequalities etc. which are called the mathematical models. Then solve the model and interpret the solution in terms of the original problem. Before we explain the process, we shall discuss what a mathematical model is.

A mathematical model is a representation which comprehends a situation.

An interesting geometric model is illustrated in the following example.

Example 2 (Bridge Problem) Konigsberg is a town on the Pregel River, which in the 18th century was a German town, but now is Russian. Within the town are two river islands that are connected to the banks with seven bridges as shown in (Fig 2).

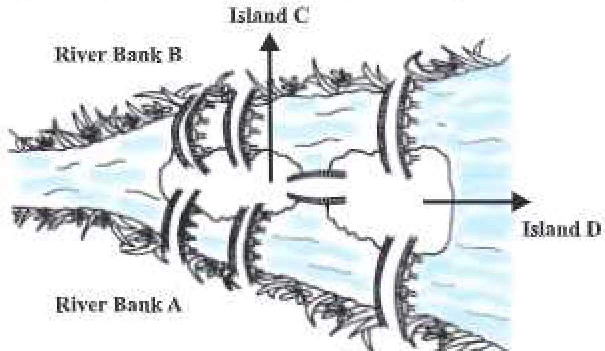


Fig 2

People tried to walk around the town in a way that only crossed each bridge once, but it proved to be difficult problem. Leonhard Euler, a Swiss mathematician in the service of the Russian empire Catherine the Great, heard about the problem. In 1736 Euler proved that the walk was not possible to do. He proved this by inventing a kind of diagram called a network, that is made up of vertices (dots where lines meet) and arcs (lines) (Fig3).

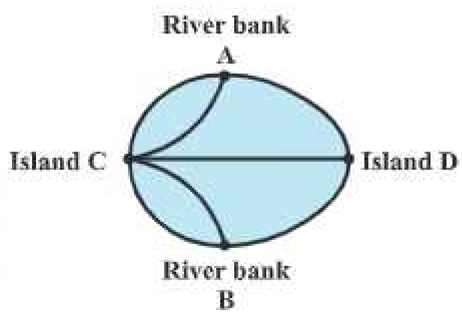


Fig 3

He used four dots (vertices) for the two river banks and the two islands. These have been marked A, B and C, D. The seven lines (arcs) are the seven bridges. You can see that 3 bridges (arcs) join to riverbank, A, and 3 join to riverbank B. 5 bridges (arcs) join to island C, and 3 join to island D. This means that all the vertices have an odd number of arcs, so they are called odd vertices (An even vertex would have to have an even number of arcs joining to it).

Remember that the problem was to travel around town crossing each bridge only once. On Euler's network this meant tracing over each arc only once, visiting all the vertices. Euler proved it could not be done because he worked out that, to have an odd vertex you would have to begin or end the trip at that vertex. (Think about it). Since there can only be one beginning and one end, there can only be two odd vertices if you are to trace over each arc only once. Since the bridge problem has 4 odd vertices, it just not possible to do!

After Euler proved his Theorem, much water has flown under the bridges in Konigsberg. In 1875, an extra bridge was built in Konigsberg, joining the land areas of river banks A and B (Fig 4). Is it possible now for the Konigsbergians to go round the city, using each bridge only once?

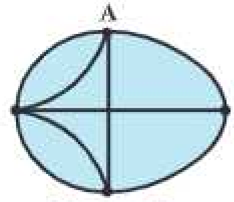


Fig 4

Here the situation will be as in Fig 4. After the addition of the new edge, both the vertices A and B have become even degree vertices. However, D and C still have odd degree. So, it is possible for the Konigsbergians to go around the city using each bridge exactly once.

The invention of networks began a new theory called graph theory which is now used in many ways, including planning and mapping railway networks (Fig 4).

A.2.3 What is Mathematical Modelling?

Here, we shall define what mathematical modelling is and illustrate the different processes involved in this through examples.

Definition Mathematical modelling is an attempt to study some part (or form) of the real-life problem in mathematical terms.

Conversion of physical situation into mathematics with some suitable conditions is known as mathematical modelling. Mathematical modelling is nothing but a technique and the pedagogy taken from fine arts and not from the basic sciences. Let us now understand the different processes involved in Mathematical Modelling. Four steps are involved in this process. As an illustrative example, we consider the modelling done to study the motion of a simple pendulum.

Understanding the problem

This involves, for example, understanding the process involved in the motion of simple pendulum. All of us are familiar with the simple pendulum. This pendulum is simply a mass (known as bob) attached to one end of a string whose other end is fixed at a point. We have studied that the motion of the simple pendulum is periodic. The period depends upon the length of the string and acceleration due to gravity. So, what we need to find is the period of oscillation. Based on this, we give a precise statement of the problem as

Statement How do we find the period of oscillation of the simple pendulum?
The next step is formulation.

Formulation Consists of two main steps.

1. Identifying the relevant factors In this, we find out what are the factors/

parameters involved in the problem. For example, in the case of pendulum, the factors are period of oscillation (T), the mass of the bob (m), effective length (l) of the pendulum which is the distance between the point of suspension to the centre of mass of the bob. Here, we consider the length of string as effective length of the pendulum and acceleration due to gravity (g), which is assumed to be constant at a place.

So, we have identified four parameters for studying the problem. Now, our purpose is to find T. For this we need to understand what are the parameters that affect the period which can be done by performing a simple experiment.

We take two metal balls of two different masses and conduct experiment with each of them attached to two strings of equal lengths. We measure the period of oscillation. We make the observation that there is no appreciable change of the period with mass. Now, we perform the same experiment on equal mass of balls but take strings of different lengths and observe that there is clear dependence of the period on the length of the pendulum.

This indicates that the mass m is not an *essential parameter* for finding period whereas the length l is an essential parameter.

This process of searching the **essential parameters** is necessary before we go to the next step.

2. Mathematical description This involves finding an equation, inequality or a geometric figure using the parameters already identified.

In the case of simple pendulum, experiments were conducted in which the values of period T were measured for different values of l. These values were plotted on a graph which resulted in a curve that resembled a parabola. It implies that the relation between T and l could be expressed

$$T^2 = kl \quad \text{... (1)}$$

It was found that $k = \frac{4\pi^2}{g}$. This gives the equation

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{... (2)}$$

Equation (2) gives the mathematical formulation of the problem.

Finding the solution The mathematical formulation rarely gives the answer directly. Usually we have to do some operation which involves solving an equation, calculation or applying a theorem etc. In the case of simple pendulums the solution involves applying the formula given in Equation (2).

The period of oscillation calculated for two different pendulums having different lengths is given in Table 1

Table 1

l	225 cm	275cm
T	3.04 sec	3.36 sec

The table shows that for l = 225 cm, T = 3.04 sec and for l = 275 cm, T = 3.36 sec.

Interpretation/Validation

A mathematical model is an attempt to study, the essential characteristic of a real life problem. Many times model equations are obtained by assuming the situation in an idealised context. The model will be useful only if it explains all the facts that we would like it to explain. Otherwise, we will reject it, or else, improve it, then test it again. In other words, ***we measure the effectiveness of the model by comparing the results obtained from the mathematical model, with the known facts about the real problem. This process is called validation of the model.*** In the case of simple pendulum, we conduct some experiments on the pendulum and find out period of oscillation. The results of the experiment are given in Table 2.

Table 2

Periods obtained experimentally for four different pendulums

Mass (gms)	Length (cms)	Time (secs)
385	275	3.371
	225	3.056
230	275	3.352
	225	3.042

Now, we compare the measured values in Table 2 with the calculated values given in Table 1.

The difference in the observed values and calculated values gives the error. For example, for l = 275 cm, and mass m = 385 gm,

$$\text{error} = 3.371 - 3.36 = 0.011$$

which is small and the model is accepted.

Once we accept the model, we have to interpret the model. ***The process of describing the solution in the context of the real situation is called interpretation of the model.*** In this case, we can interpret the solution in the following way:

(a) The period is directly proportional to the square root of the length of the pendulum.

(b) It is inversely proportional to the square root of the acceleration due to gravity.

Our validation and interpretation of this model shows that the mathematical model is in good agreement with the practical (or observed) values. But we found that there is some error in the calculated result and measured result. This is because we have neglected the mass of the string and resistance of the medium. So, in such situation we look for a better model and this process continues.

This leads us to an important observation. The real world is far too complex to understand and describe completely. We just pick one or two main factors to be completely accurate that may influence the situation. Then try to obtain a simplified model which gives some information about the situation. We study the simple situation with this model expecting that we can obtain a better model of the situation.
Now, we summarise the main process involved in the modelling as

(a) Formulation (b) Solution (c) Interpretation/Validation

The next example shows how modelling can be done using the techniques of finding graphical solution of inequality.

Example 3 A farm house uses atleast 800 kg of special food daily. The special food is a mixture of corn and soyabean with the following compositions

Table 3

Material	Nutrients present per Kg Protein	Nutrients present per Kg Fibre	Cost per Kg
Corn	.09	.02	Rs 10
Soyabean	.60	.06	Rs 20

The dietary requirements of the special food stipulate atleast 30% protein and at most 5% fibre. Determine the daily minimum cost of the food mix.

Solution Step 1 Here the objective is to minimise the total daily cost of the food which is made up of corn and soyabean. So the variables (factors) that are to be considered are

x = the amount of corn

y = the amount of soyabean

z = the cost

Step 2 The last column in Table 3 indicates that z, x, y are related by the equation

$$z = 10x + 20y \qquad \text{... (1)}$$

The problem is to minimise z with the following constraints:

(a) The farm used atleast 800 kg food consisting of corn and soyabean

i.e., $x + y \geq 800$ ... (2)

(b) The food should have atleast 30% protein dietary requirement in the proportion as given in the first column of Table 3. This gives

$$0.09x + 0.6y \geq 0.3\,(x + y) \qquad \text{... (3)}$$

(c) Similarly the food should have atmost 5% fibre in the proportion given in 2nd column of Table 3. This gives

$$0.02x + 0.06\,y \leq 0.05\,(x + y) \qquad \text{... (4)}$$

We simplify the constraints given in (2), (3) and (4) by grouping all the coefficients of x, y.

Then the problem can be restated in the following mathematical form.
Statement Minimise z subject to

$$x + y \geq 800$$
$$0.21x - .30y \leq 0$$
$$0.03x - .01y \geq 0$$

This gives the formulation of the model.

Step 3 This can be solved graphically. The shaded region in Fig 5 gives the possible solution of the equations. From the graph it is clear that the minimum value is got at the point (470.6,329.4) i.e., $x = 470.6$ and $y = 329.4$.

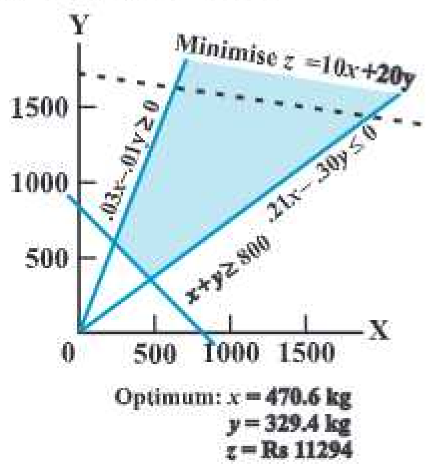


Fig 5

This gives the value of z as $z = 10 \times 470.6 + 20 \times 329.4 = 11294$

This is the mathematical solution.

Step 4 The solution can be interpreted as saying that, "The minimum cost of the special food with corn and soyabean having the required portion of nutrient contents, protein and fibre is Rs 11294 and we obtain this minimum cost if we use 470.6 kg of corn and 329.4 kg of soyabean."

In the next example, we shall discuss how modelling is used to study the population of a country at a particular time.

Example 4 Suppose a population control unit wants to find out "how many people will be there in a certain country after 10 years"

Step 1 Formulation We first observe that the population changes with time and it increases with birth and decreases with deaths.

We want to find the population at a particular time. Let t denote the time in years. Then t takes values 0, 1, 2, ..., $t = 0$ stands for the present time, $t = 1$ stands for the next year etc. For any time t, let $p(t)$ denote the population in that particular year.

Suppose we want to find the population in a particular year, say $t_0 = 2006$. How will we do that. We find the population by Jan. 1st, 2005. Add the number of births in that year and subtract the number of deaths in that year. Let B(t) denote the number of births in the one year between t and $t + 1$ and D(t) denote the number of deaths between t and $t + 1$. Then we get the relation

$$P(t+1) = P(t) + B(t) - D(t)$$

Now we make some assumptions and definitions

1. $\frac{B(t)}{P(t)}$ is called the *birth rate* for the time interval t to $t + 1$.

2. $\frac{D(t)}{P(t)}$ is called the *death rate* for the time interval t to $t + 1$.

Assumptions

1. The birth rate is the same for all intervals. Likewise, the death rate is the same for all intervals. This means that there is a constant b, called the birth rate, and a constant d, called the death rate so that, for all $t \geq 0$,

$$b = \frac{B(t)}{P(t)} \quad \text{and} \quad d = \frac{D(t)}{P(t)} \qquad \dots (1)$$

2. There is no migration into or out of the population; i.e., the only source of population change is birth and death.

As a result of assumptions 1 and 2, we deduce that, for $t \geq 0$,

$$\begin{aligned} P(t+1) &= P(t) + B(t) - D(t) \\ &= P(t) + bP(t) - dP(t) \\ &= (1 + b - d)\, P(t) \end{aligned} \quad \text{... (2)}$$

Setting $t = 0$ in (2) gives

$$P(1) = (1 + b - d)P(0) \quad \text{... (3)}$$

Setting $t = 1$ in Equation (2) gives

$$\begin{aligned} P(2) &= (1 + b - d)\, P(1) \\ &= (1 + b - d)\,(1 + b - d)\, P(0) \qquad \text{(Using equation 3)} \\ &= (1 + b - d)^2\, P(0) \end{aligned}$$

Continuing this way, we get

$$P(t) = (1 + b - d)^t\, P(0) \quad \text{... (4)}$$

for $t = 0, 1, 2, ...$ The constant $1 + b - d$ is often abbreviated by r and called the *growth rate or*, in more high-flown language, the *Malthusian parameter*, in honor of Robert Malthus who first brought this model to popular attention. In terms of r, Equation (4) becomes

$$P(t) = P(0)r^t, \qquad t = 0, 1, 2, ... \quad \text{... (5)}$$

$P(t)$ is an example of an *exponential function*. Any function of the form cr^t, where c and r are constants, is an exponential function.

Equation (5) gives the mathematical formulation of the problem.

Step 2 – Solution

Suppose the current population is 250,000,000 and the rates are $b = 0.02$ and $d = 0.01$. What will the population be in 10 years? Using the formula, we calculate P(10).

$$\begin{aligned} P(10) &= (1.01)^{10}\,(250{,}000{,}000) \\ &= (1.104622125)\,(250{,}000{,}000) \\ &= 276{,}155{,}531.25 \end{aligned}$$

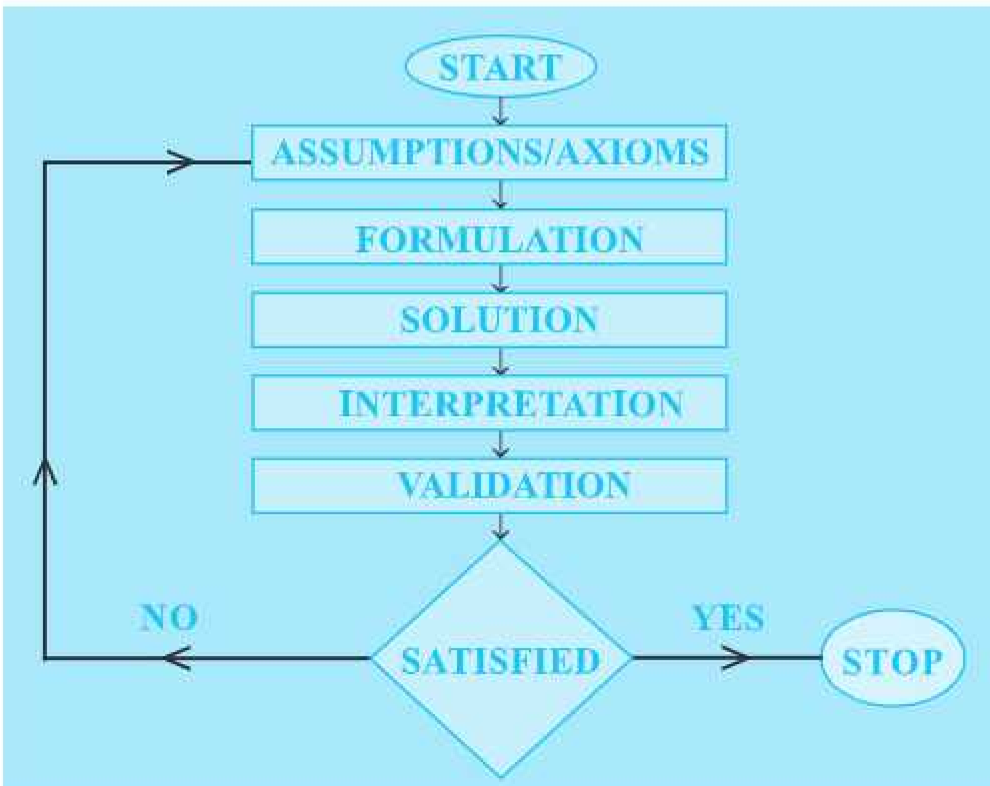
Step 3 Interpretation and Validation

Naturally, this result is absurd, since one can't have 0.25 of a person.

So, we do some approximation and conclude that the population is 276,155,531 (approximately). Here, we are not getting the exact answer because of the assumptions that we have made in our mathematical model.

The above examples show how modelling is done in variety of situations using different mathematical techniques.

Since a mathematical model is a simplified representation of a real problem, by its very nature, has built-in assumptions and approximations. Obviously, the most important question is to decide whether our model is a good one or not i.e., when the obtained results are interpreted physically whether or not the model gives reasonable answers. If a model is not accurate enough, we try to identify the sources of the shortcomings. It may happen that we need a new formulation, new mathematical manipulation and hence a new evaluation. Thus mathematical modelling can be a cycle of the modelling process as shown in the flowchart given below:



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ANSWERS

EXERCISE 1.1

1. (i), (iv), (v), (vi), (vii) and (viii) are sets.
2. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (vi) $\in$ (v) $\in$ (vi) $\notin$
3. (i) A = {–3, –2, –1, 0, 1, 2, 3, 4, 5, 6 } (ii) B = {1, 2, 3, 4, 5}
   (iii) C = {17, 26, 35, 44, 53, 62, 71, 80} (iv) D = {2, 3, 5}
   (v) E = {T, R, I, G, O, N, M, E, Y} (vi) F = {B, E, T, R}
4. (i) $\{ x : x = 3n, n \in \mathbf{N} \text{ and } 1 \le n \le 4 \}$ (ii) $\{ x : x = 2^n, n \in \mathbf{N} \text{ and } 1 \le n \le 5 \}$
   (iii) $\{ x : x = 5^n, n \in \mathbf{N} \text{ and } 1 \le n \le 4 \}$ (iv) $\{ x : x$ is an even natural number$\}$
   (v) $\{ x : x = n^2, n \in \mathbf{N} \text{ and } 1 \le n \le 10 \}$
5. (i) A = {1, 3, 5, . . . } (ii) B = {0, 1, 2, 3, 4 }
   (iii) C = { –2, –1, 0, 1, 2 } (iv) D = { L, O, Y, A }
   (v) E = { February, April, June, September, November }
   (vi) F = $\{b, c, d, f, g, h, j \}$
6. (i) $\leftrightarrow$ (c) (ii) $\leftrightarrow$ (a) (iii) $\leftrightarrow$ (d) (iv) $\leftrightarrow$ (b)

EXERCISE 1.2

1. (i), (iii), (iv)
2. (i) Finite (ii) Infinite (iii) Finite (iv) Infinite (v) Finite
3. (i) Infinite (ii) Finite (iii) Infinite (iv) Finite (v) Infinite
4. (i) Yes (ii) No (iii) Yes (iv) No
5. (i) No (ii) Yes 6. B= D, E = G

EXERCISE 1.3

1. (i) $\subset$ (ii) $\not\subset$ (iii) $\subset$ (iv) $\not\subset$ (v) $\not\subset$ (vi) $\subset$
   (vii) $\subset$
2. (i) False (ii) True (iii) False (iv) True (v) False (vi) True
3. (i) as $\{3,4\} \in$ A, (v) as $1 \in$ A, (vii) as $\{1, 2, 5\} \subset$ A,
   (viii) as $3 \notin$ A, (ix) as $\phi \subset$ A, (xi) as $\phi \subset$ A,
4. (i) ϕ, $\{ a \}$ (ii) ϕ, $\{ a \}$, $\{ b \}$, $\{ a, b \}$
   (iii) ϕ, { 1 }, { 2 }, { 3 }, { 1, 2 }, { 1, 3 }, { 2, 3 }, { 1, 2, 3 } (iv) ϕ
5. (i) $(-4, 6]$ (ii) $(-12, -10)$ (iii) $[0, 7)$
   (iv) $[3, 4]$
6. (i) $\{ x : x \in \mathbf{R}, -3 < x < 0 \}$ (ii) $\{ x : x \in \mathbf{R}, 6 \le x \le 12 \}$
   (iii) $\{ x : x \in \mathbf{R}, 6 < x \le 12 \}$ (iv) $\{ x\ \mathbf{R} : -23 \le x < 5 \}$ 8. (iii)

EXERCISE 1.4

1. (i) $X \cup Y = \{1, 2, 3, 5\}$ (ii) $A \cup B = \{a, b, c, e, i, o, u\}$

(iii) $A \cup B = \{x : x = 1, 2, 4, 5$ or a multiple of $3\}$

(iv) $A \cup B = \{x : 1 < x < 10, x \in N\}$ (v) $A \cup B = \{1, 2, 3\}$

2. Yes, $A \cup B = \{a, b, c\}$ **3.** B

4. (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (iii) $\{3, 4, 5, 6, 7, 8\}$

(iv) $\{3, 4, 5, 6, 7, 8, 9, 10\}$ (v) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

(vi) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (vii) $\{3, 4, 5, 6, 7, 8, 9, 10\}$

5. (i) $X \cap Y = \{1, 3\}$ (ii) $A \cap B = \{a\}$ (iii) $\{3\}$ (iv) ϕ (v) ϕ

6. (i) $\{7, 9, 11\}$ (ii) $\{11, 13\}$ (iii) ϕ (iv) $\{11\}$

(v) ϕ (vi) $\{7, 9, 11\}$ (vii) ϕ

(viii) $\{7, 9, 11\}$ (ix) $\{7, 9, 11\}$ (x) $\{7, 9, 11, 15\}$

7. (i) B (ii) C (iii) D (iv) ϕ

(v) $\{2\}$ (vi) $\{x : x$ is an odd prime number $\}$ **8.** (iii)

9. (i) $\{3, 6, 9, 15, 18, 21\}$ (ii) $\{3, 9, 15, 18, 21\}$ (iii) $\{3, 6, 9, 12, 18, 21\}$

(iv) $\{4, 8, 16, 20\}$ (v) $\{2, 4, 8, 10, 14, 16\}$ (vi) $\{5, 10, 20\}$

(vii) $\{20\}$ (viii) $\{4, 8, 12, 16\}$ (ix) $\{2, 6, 10, 14\}$

(x) $\{5, 10, 15\}$ (xi) $\{2, 4, 6, 8, 12, 14, 16\}$ (xii) $\{5, 15, 20\}$

10. (i) $\{a, c\}$ (ii) $\{f, g\}$ (iii) $\{b, d\}$

11. Set of irrational numbers **12.** (i) F (ii) F (iii) T (iv) T

EXERCISE 1.5

1. (i) $\{5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9\}$ (iii) $\{7, 8, 9\}$

(iv) $\{5, 7, 9\}$ (v) $\{1, 2, 3, 4\}$ (vi) $\{1, 3, 4, 5, 6, 7, 9\}$

2. (i) $\{d, e, f, g, h\}$ (ii) $\{a, b, c, h\}$ (iii) $\{b, d, f, h\}$

(iv) $\{b, c, d, e\}$

3. (i) $\{x : x$ is an odd natural number $\}$

(ii) $\{x : x$ is an even natural number $\}$

(iii) $\{x : x \in \mathbf{N}$ and x is not a multiple of $3\}$

(iv) { $x : x$ is a positive composite number or $x = 1$ }
(v) { $x : x$ is a positive integer which is not divisible by 3 or not divisible by 5}
(vi) { $x : x \in \mathbf{N}$ and x is not a perfect square }
(vii) { $x : x \in \mathbf{N}$ and x is not a perfect cube }
(viii) { $x : x \in \mathbf{N}$ and $x \neq 3$ } (ix) { $x : x \in \mathbf{N}$ and $x \neq 2$ }
(x) { $x : x \in \mathbf{N}$ and $x < 7$ } (xi) { $x : x \in \mathbf{N}$ and $x \leq \frac{9}{2}$}

6. A' is the set of all equilateral triangles.
7. (i) U (ii) A (iii) ϕ (iv) ϕ

Miscellaneous Exercise on Chapter 1

1. $A \subset B$, $A \subset C$, $B \subset C$, $D \subset A$, $D \subset B$, $D \subset C$
2. (i) False (ii) False (iii) True (iv) False (v) False
(vi) True
10. We may take A = { 1, 2 }, B = { 1, 3 }, C = { 2 , 3 }

EXERCISE 2.1

1. $x = 2$ and $y = 1$ 2. The number of elements in A × B is 9.
3. G × H = {(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)}
H × G = {(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)}
4. (i) False
P × Q = {(m, n), (m, m), (n, n), (n, m)}
(ii) True
(iii) True
5. A × A = {(– 1, – 1), (– 1, 1), (1, – 1), (1, 1)}
A × A × A = {(–1, –1, –1), (–1, –1, 1), (–1, 1, –1), (–1, 1, 1), (1, –1, –1), (1, –1, 1), (1, 1, –1), (1, 1, 1)}
6. A = {a, b}, B = {x, y}
8. A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}
A × B will have $2^4 = 16$ subsets.
9. A = {x, y, z} and B = {1,2}
10. A = {–1, 0, 1}, remaining elements of
A × A are (–1, –1), (–1, 1), (0, –1), (0, 0), (1, –1), (1, 0), (1, 1)

EXERCISE 2.2

1. R = {(1, 3), (2, 6), (3, 9), (4, 12)}
   Domain of R = {1, 2, 3, 4}
   Range of R = {3, 6, 9, 12}
   Co domain of R = {1, 2, ..., 14}
2. R = {(1, 6), (2, 7), (3, 8)}
   Domain of R = {1, 2, 3}
   Range of R = {6, 7, 8}
3. R = {(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)}
4. (i) R = $\{(x, y) : y = x - 2$ for $x = 5, 6, 7\}$
   (ii) R = {(5,3), (6,4), (7,5)}. Domain of R = {5, 6, 7}, Range of R = {3, 4, 5}
5. (i) R = {(1, 1), (1,2), (1, 3), (1, 4), (1, 6), (2 4), (2, 6), (2, 2), (4, 4), (6, 6), (3, 3), (3, 6)}
   (ii) Domain of R = {1, 2, 3, 4, 6}
   (iii) Range of R = {1, 2, 3, 4, 6}
6. Domain of R = {0, 1, 2, 3, 4, 5,}
   Range of R = {5, 6, 7, 8, 9, 10}
7. R = {(2, 8), (3, 27), (5, 125), (7, 343)}
8. No. of relations from A into B = 2^6
9. Domain of R = **Z**
   Range of R = **Z**

EXERCISE 2.3

1. (i) yes, Domain = {2, 5, 8, 11, 14, 17}, Range = {1}
   (ii) yes, Domain = (2, 4, 6, 8, 10, 12, 14}, Range = {1, 2, 3, 4, 5, 6, 7}
   (iii) No.
2. (i) Domain = **R**, Range = $(-\infty, 0]$
   (ii) Domain of function = $\{x : -3 \leq x \leq 3\}$
   Range of function = $\{x : 0 \leq x \leq 3\}$
3. (i) $f(0) = -5$ (ii) $f(7) = 9$ (iii) $f(-3) = -11$
4. (i) $t(0) = 32$ (ii) $t(28) = \dfrac{412}{5}$ (iii) $t(-10) = 14$ (iv) 100
5. (i) Range = $(-\infty, 2)$ (ii) Range = $[2, \infty)$ (iii) Range = **R**

Miscellaneous Exercise on Chapter 2

2. 2.1 **3.** Domain of function is set of real numbers except 6 and 2.

4. Domain $= [1, \infty)$, Range $= [0, \infty)$

5. Domain $= \mathbf{R}$, Range = non-negative real numbers

6. Range $= [0, 1)$

7. $(f + g)\, x = 3x - 2$ **8.** $a = 2,\ b = -1$ **9.** (i) No (ii) No (iii) No

$(f - g)\, x = -x + 4$

$\left(\frac{f}{g}\right) x = \frac{x+1}{2x-3},\ \ x \neq \frac{3}{2}$

10. (i) Yes, (ii) No **11.** No **12.** Range of $f = \{3, 5, 11, 13\}$

EXERCISE 3.1

1. (i) $\frac{5\pi}{36}$ (ii) $-\frac{19\pi}{72}$ (iii) $\frac{4\pi}{3}$ (iv) $\frac{26\pi}{9}$

2. (i) $39°\ 22'\ 30''$ (ii) $-229°\ 5'\ 27''$ (iii) $300°$ (iv) $210°$

3. 12π **4.** $12°\ 36'$ **5.** $\frac{20\pi}{3}$ **6.** 5 : 4

7. (i) $\frac{2}{15}$ (ii) $\frac{1}{5}$ (iii) $\frac{7}{25}$

EXERCISE 3.2

1. $\sin x = -\frac{\sqrt{3}}{2}$, $\operatorname{cosec} x = -\frac{2}{\sqrt{3}}$, $\sec x = -2$, $\tan x = \sqrt{3}$, $\cot x = \frac{1}{\sqrt{3}}$

2. $\operatorname{cosec} x = \frac{5}{3}$, $\cos x = -\frac{4}{5}$, $\sec x = -\frac{5}{4}$, $\tan x = -\frac{3}{4}$, $\cot x = -\frac{4}{3}$

3. $\sin x = -\frac{4}{5}$, $\operatorname{cosec} x = -\frac{5}{4}$, $\cos x = -\frac{3}{5}$, $\sec x = -\frac{5}{3}$, $\tan x = \frac{4}{3}$

4. $\sin x = -\frac{12}{13}$, $\operatorname{cosec} x = -\frac{13}{12}$, $\cos x = \frac{5}{13}$, $\tan x = -\frac{12}{5}$, $\cot x = -\frac{5}{12}$

5. $\sin x = \frac{5}{13}$, $\operatorname{cosec} x = \frac{13}{5}$, $\cos x = -\frac{12}{13}$, $\sec x = -\frac{13}{12}$, $\cot x = -\frac{12}{5}$

6. $\frac{1}{\sqrt{2}}$ 7. 2 8. $\sqrt{3}$ 9. $\frac{\sqrt{3}}{2}$ 10. 1

EXERCISE 3.3

5. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $2-\sqrt{3}$

Miscellaneous Exercise on Chapter 3

8. $\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, \frac{1}{2}$

9. $\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}, -\sqrt{2}$

10. $\frac{\sqrt{8+2\sqrt{15}}}{4}, \frac{\sqrt{8-2\sqrt{15}}}{4}, 4+\sqrt{15}$

EXERCISE 4.1

1. $3 + i0$ 2. $0 + i\,0$ 3. $0+i\,1$ 4. $14 + 28\,i$

5. $2 - 7\,i$ 6. $-\frac{19}{5}-\frac{21i}{10}$ 7. $\frac{17}{3}+i\frac{5}{3}$ 8. $-4 + i\,0$

9. $-\frac{242}{27}-26i$ 10. $\frac{-22}{3}-i\frac{107}{27}$ 11. $\frac{4}{25}+i\frac{3}{25}$ 12. $\frac{\sqrt{5}}{14}-i\frac{3}{14}$

13. $0 + i1$ 14. $0 - i\,\frac{7\sqrt{2}}{2}$

Miscellaneous Exercise on Chapter 4

1. $2-2i$ **3.** $\dfrac{307+599i}{442}$

5. $\sqrt{2}$ **7.** (i) $\dfrac{-2}{5}$, (ii) 0 **8.** $x = 3, y = -3$ **9.** 2

11. 1 **12.** 0 **14.** 4

EXERCISE 5.1

1. (i) {1, 2, 3, 4} (ii) {... – 3, – 2, –1, 0, 1,2,3,4,}

2. (i) No Solution (ii) {... – 4, – 3}

3. (i) {... – 2, – 1, 0, 1} (ii) $(-\infty, 2)$

4. (i) {–1, 0, 1, 2, 3, ...} (ii) $(-2, \infty)$

5. $(-4, \infty)$ **6.** $(-\infty, -3)$ **7.** $(-\infty, -3]$ **8.** $(-\infty, 4]$

9. $(-\infty, 6)$ **10.** $(-\infty, -6)$ **11.** $(-\infty, 2]$ **12.** $(-\infty, 120]$

13. $(4, \infty)$ **14.** $(-\infty, 2]$ **15.** $(4, \infty)$ **16.** $(-\infty, 2]$

17. $(-\infty, 3)$,


18. $[-1, \infty)$,


19. $(-1, \infty)$,


20. $\left[-\dfrac{2}{7}, \infty\right)$,


21. 35 **22.** 82

23. (5,7), (7,9) **24.** (6,8), (8,10), (10,12)

25. 9 cm **26.** Greater than or equal to 8cm but less than or equal to 22cm

Miscellaneous Exercise on Chapter 5

1. [2, 3] **2.** (0, 1] **3.** [– 4, 2]

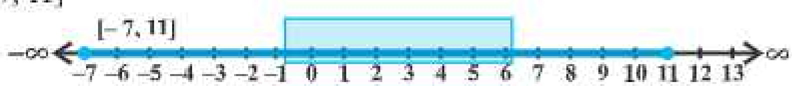
4. (– 23, 2] **5.** $\left(\frac{-80}{3}, \frac{-10}{3}\right]$ **6.** $\left[1, \frac{11}{3}\right]$

7. (–5, 5)

8. (–1, 7)

9. (5, ∞)

10. [– 7, 11]



11. Between 20°C and 25°C

12. More than 320 litres but less than 1280 litres.

13. More than 562.5 litres but less than 900 litres.

14. 9.6 ≤ MA ≤ 16.8

EXERCISE 6.1

1. (i) 125, (ii) 60. **2.** 108 **3.** 5040 **4.** 336

5. 8 **6.** 20

EXERCISE 6.2

1. (i) 40320, (ii) 18 **2.** 30, No **3.** 28 **4.** 64

5. (i) 30, (ii) 15120

EXERCISE 6.3

1. 504 **2.** 4536 **3.** 60 **4.** 120, 48

5. 56 **6.** 9 **7.** (i) 3, (ii) 4 **8.** 40320

9. (i) 360, (ii) 720, (iii) 240
10. 33810
11. (i) 1814400, (ii) 2419200, (iii) 25401600

EXERCISE 6.4

1. 45
2. (i) 5, (ii) 6
3. 210
4. 40
5. 2000
6. 778320
7. 3960
8. 200
9. 35

Miscellaneous Exercise on Chapter 6

1. 3600
2. 1440
3. (i) 504, (ii) 588, (iii) 1632
4. 907200
5. 120
6. 50400
7. 420
8. ${}^{4}C_1 \times {}^{48}C_4$
9. 2880
10. ${}^{22}C_7 + {}^{22}C_{10}$
11. 151200

EXERCISE 7.1

1. $1-10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$
2. $\frac{32}{x^5}-\frac{40}{x^3}+\frac{20}{x}-5x+\frac{5}{8}x^3-\frac{x^5}{32}$
3. $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$
4. $\frac{x^5}{243}+\frac{5x^3}{81}+\frac{10}{27}x+\frac{10}{9x}+\frac{5}{3x^3}+\frac{1}{x^5}$
5. $x^6+6x^4+15x^2+20+\frac{15}{x^2}+\frac{6}{x^4}+\frac{1}{x^6}$
6. 884736
7. 11040808032
8. 104060401
9. 9509900499
10. $(1.1)^{10000} > 1000$
11. $8(a^3b + ab^3)$; $40\sqrt{6}$
12. $2(x^6 + 15x^4 + 15x^2 + 1)$, 198

Miscellaneous Exercise on Chapter 7

2. $396\sqrt{6}$
3. $2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$
4. 0.9510
5. $\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-5$
6. $27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

EXERCISE 8.1

1. 3, 8, 15, 24, 35
2. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$
3. 2, 4, 8, 16 and 32
4. $-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$ and $\frac{7}{6}$
5. 25, –125, 625, –3125, 15625
6. $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}, 21$ and $\frac{75}{2}$
7. 65, 93
8. $\frac{49}{128}$
9. 729
10. $\frac{360}{23}$
11. 3, 11, 35, 107, 323; 3 + 11 + 35 + 107 + 323 + ...
12. $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}, \frac{-1}{120}; -1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{6}\right) + \left(\frac{-1}{24}\right) + \left(\frac{-1}{120}\right) + ...$
13. 2, 2, 1, 0, –1; 2 + 2 + 1 + 0 + (–1) + ...
14. $1, 2, \frac{3}{2}, \frac{5}{3}$ and $\frac{8}{5}$

EXERCISE 8.2

1. $\frac{5}{2^{20}}, \frac{5}{2^n}$
2. 3072
4. -2187
5. (a) 13th, (b) 12th, (c) 9th
6. ± 1
7. $\frac{1}{6}\left[1-(0.1)^{20}\right]$
8. $\frac{\sqrt{7}}{2}\left(\sqrt{3}+1\right)\left(3^{\frac{n}{2}}-1\right)$
9. $\frac{\left[1-(-a)^n\right]}{1+a}$
10. $\frac{x^3\left(1-x^{2n}\right)}{1-x^2}$
11. $22+\frac{3}{2}\left(3^{11}-1\right)$
12. $r=\frac{5}{2}$ or $\frac{2}{5}$; Terms are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$
13. 4
14. $\frac{16}{7}; 2; \frac{16}{7}\left(2^n-1\right)$
15. 2059 or 463

16. $\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, ...$ or $4, -8, 16, -32, 64, ..$ **18.** $\frac{80}{81}\left(10^n - 1\right) - \frac{8}{9}n$

19. 496 **20.** rR **21.** 3, –6, 12, –24 **26.** 9 and 27

27. $n = \frac{-1}{2}$ **30.** 120, 480, 30 (2^n) **31.** Rs 500 $(1.1)^{10}$

32. $x^2 - 16x + 25 = 0$

Miscellaneous Exercise on Chapter 8

1. 4 **2.** 160; 6 **3.** ± 3 **4.** 8, 16, 32

5. 4 **11.** (i) $\frac{50}{81}\left(10^n - 1\right) - \frac{5n}{9}$, (ii) $\frac{2n}{3} - \frac{2}{27}\left(1 - 10^{-n}\right)$

12. 1680

13. Rs 16680 **14.** Rs 39100 **15.** Rs 43690 **16.** Rs 17000; 20,000

17. Rs 5120 **18.** 25 days

EXERCISE 9.1

1. $\frac{121}{2}$ square unit.

2. $(0, a)$, $(0, -a)$ and $\left(-\sqrt{3}a, 0\right)$ or $(0, a)$, $(0, -a)$, and $\left(\sqrt{3}a, 0\right)$

3. (i) $|y_2 - y_1|$, (ii) $|x_2 - x_1|$ **4.** $\left(\frac{15}{2}, 0\right)$ **5.** $-\frac{1}{2}$

7. $-\sqrt{3}$ **9.** 135°

10. 1 and 2, or $\frac{1}{2}$ and 1, or – 1 and –2, or $-\frac{1}{2}$ and – 1

EXERCISE 9.2

1. $y = 0$ and $x = 0$ **2.** $x - 2y + 10 = 0$ **3.** $y = mx$

4. $\left(\sqrt{3}+1\right)x - \left(\sqrt{3}-1\right)y = 4\left(\sqrt{3}-1\right)$ **5.** $2x + y + 6 = 0$

6. $x - \sqrt{3}y + 2\sqrt{3} = 0$ **7.** $5x + 3y + 2 = 0$

8. $3x - 4y + 8 = 0$
9. $5x - y + 20 = 0$
10. $(1 + n)x + 3(1 + n)y = n + 11$
11. $x + y = 5$
12. $x + 2y - 6 = 0,\ 2x + y - 6 = 0$
13. $\sqrt{3}x + y - 2 = 0$ and $\sqrt{3}x + y + 2 = 0$
14. $2x - 9y + 85 = 0$
15. $\text{L} = \frac{.192}{90}(\text{C} - 20) + 124.942$
16. 1340 litres.
18. $2kx + hy = 3kh$.

EXERCISE 9.3

1. (i) $y = -\frac{1}{7}x + 0, -\frac{1}{7}, 0$; (ii) $y = -2x + \frac{5}{3}, -2, \frac{5}{3}$; (iii) $y = 0x + 0, 0, 0$

2. (i) $\frac{x}{4} + \frac{y}{6} = 1, 4, 6$; (ii) $\frac{x}{\frac{3}{2}} + \frac{y}{-2} = 1, \frac{3}{2}, -2$;

   (iii) $y = -\frac{2}{3}$, intercept with y-axis $= -\frac{2}{3}$ and no intercept with x-axis.

3. 5 units

4. $(-2, 0)$ and $(8, 0)$
5. (i) $\frac{65}{17}$ units, (ii) $\frac{1}{\sqrt{2}}\left|\frac{p+r}{l}\right|$ units.
6. $3x - 4y + 18 = 0$
7. $y + 7x = 21$
8. 30° and 150°
9. $\frac{22}{9}$
11. $(\sqrt{3} + 2)x + (2\sqrt{3} - 1)y = 8\sqrt{3} + 1$ or $(\sqrt{3} - 2)x + (1 + 2\sqrt{3})y = -1 + 8\sqrt{3}$
12. $2x + y = 5$
13. $\left(\frac{68}{25}, -\frac{49}{25}\right)$
14. $m = \frac{1}{2}, c = \frac{5}{2}$
16. $y - x = 1,\ \sqrt{2}$

Miscellaneous Exercise on Chapter 9

1. (a) 3, (b) ± 2, (c) 6 or 1
2. $2x - 3y = 6, -3x + 2y = 6$
3. $\left(0, -\frac{8}{3}\right), \left(0, \frac{32}{3}\right)$

4. $\left|\cos\dfrac{\phi-\theta}{2}\right|$ 5. $x=-\dfrac{5}{22}$ 6. $2x-3y+18=0$

7. k^2 square units 8. 5 10. $3x-y=7,\ \ x+3y=9$

11. $13x+13y=6$ 13. 1 : 2 14. $\dfrac{23\sqrt{5}}{18}$ units

15. The line is parallel to x - axis or parallel to y-axis

16. $x=1,\ \ y=1$ *or* $x=-4,\ \ y=3$ 17. $(-1,-4)$.

18. $\dfrac{1\pm5\sqrt{2}}{7}$ 20. $18x+12y+11=0$

21. $\left(\dfrac{13}{5},0\right)$ 23. $119x+102y=125$

EXERCISE 10.1

1. $x^2+y^2-4y=0$ 2. $x^2+y^2+4x-6y-3=0$

3. $36x^2+36y^2-36x-18y+11=0$ 4. $x^2+y^2-2x-2y=0$

5. $x^2+y^2+2ax+2by+2b^2=0$ 6. $c(-5, 3), r=6$

7. $c(2, 4), r=\sqrt{65}$ 8. $c(4,-5), r=\sqrt{53}$ 9. $c\left(\dfrac{1}{4}, 0\right); r=\dfrac{1}{4}$

10. $x^2+y^2-6x-8y+15=0$ 11. $x^2+y^2-7x+5y-14=0$

12. $x^2+y^2+4x-21=0$ & $x^2+y^2-12x+11=0$

13. $x^2+y^2-ax-by=0$ 14. $x^2+y^2-4x-4y=5$

15. Inside the circle; since the distance of the point to the centre of the circle is less than the radius of the circle.

EXERCISE 10.2

1. F (3, 0), axis - x - axis, directrix $x=-3$, length of the Latus rectum = 12

2. F $\left(0, \dfrac{3}{2}\right)$, axis - y - axis, directrix $y=-\dfrac{3}{2}$, length of the Latus rectum = 6

3. F (–2, 0), axis - x - axis, directrix $x=2$, length of the Latus rectum = 8

4. F (0, –4), axis - y - axis, directrix $y = 4$, length of the Latus rectum = 16

5. F $(\frac{5}{2}, 0)$ axis - x - axis, directrix $x = -\frac{5}{2}$, length of the Latus rectum = 10

6. F $(0, \frac{-9}{4})$, axis - y - axis, directrix $y = \frac{9}{4}$, length of the Latus rectum = 9

7. $y^2 = 24x$ 8. $x^2 = -12y$ 9. $y^2 = 12x$

10. $y^2 = -8x$ 11. $2y^2 = 9x$ 12. $2x^2 = 25y$

EXERCISE 10.3

1. F $(\pm\sqrt{20}, 0)$; V $(\pm 6, 0)$; Major axis = 12; Minor axis = 8, $e = \frac{\sqrt{20}}{6}$,

   Latus rectum = $\frac{16}{3}$

2. F $(0, \pm\sqrt{21})$; V $(0, \pm 5)$; Major axis = 10; Minor axis = 4, $e = \frac{\sqrt{21}}{5}$;

   Latus rectum = $\frac{8}{5}$

3. F $(\pm\sqrt{7}, 0)$; V $(\pm 4, 0)$; Major axis = 8; Minor axis = 6, $e = \frac{\sqrt{7}}{4}$;

   Latus rectum = $\frac{9}{2}$

4. F $(0, \pm\sqrt{75})$; V $(0, \pm 10)$; Major axis = 20; Minor axis = 10, $e = \frac{\sqrt{3}}{2}$;

   Latus rectum = 5

5. F $(\pm\sqrt{13}, 0)$; V $(\pm 7, 0)$; Major axis =14 ; Minor axis = 12, $e = \frac{\sqrt{13}}{7}$;

   Latus rectum = $\frac{72}{7}$

6. F $(0, \pm 10\sqrt{3})$; V $(0, \pm 20)$; Major axis =40 ; Minor axis = 20, $e = \frac{\sqrt{3}}{2}$;

   Latus rectum = 10

7. F $(0, \pm 4\sqrt{2})$; V $(0, \pm 6)$; Major axis $=12$; Minor axis $= 4$, $e = \frac{2\sqrt{2}}{3}$;

Latus rectum $= \frac{4}{3}$

8. $F\left(0, \pm\sqrt{15}\right)$; V $(0, \pm 4)$; Major axis $= 8$; Minor axis $= 2$, $e = \frac{\sqrt{15}}{4}$;

Latus rectum $= \frac{1}{2}$

9. F $(\pm\sqrt{5}, 0)$; V $(\pm 3, 0)$; Major axis $= 6$; Minor axis $= 4$, $e = \frac{\sqrt{5}}{3}$;

Latus rectum $= \frac{8}{3}$

10. $\frac{x^2}{25}+\frac{y^2}{9}=1$ 11. $\frac{x^2}{144}+\frac{y^2}{169}=1$ 12. $\frac{x^2}{36}+\frac{y^2}{20}=1$

13. $\frac{x^2}{9}+\frac{y^2}{4}=1$ 14. $\frac{x^2}{1}+\frac{y^2}{5}=1$ 15. $\frac{x^2}{169}+\frac{y^2}{144}=1$

16. $\frac{x^2}{64}+\frac{y^2}{100}=1$ 17. $\frac{x^2}{16}+\frac{y^2}{7}=1$ 18. $\frac{x^2}{25}+\frac{y^2}{9}=1$

19. $\frac{x^2}{10}+\frac{y^2}{40}=1$ 20. $x^2 + 4y^2 = 52$ or $\frac{x^2}{52}+\frac{y^2}{13}=1$

EXERCISE 10.4

1. Foci $(\pm 5, 0)$, Vertices $(\pm 4, 0)$; $e = \frac{5}{4}$; Latus rectum $= \frac{9}{2}$
2. Foci (0 ± 6), Vertices $(0, \pm 3)$; $e = 2$; Latus rectum $= 18$
3. Foci $(0, \pm\sqrt{13})$, Vertices $(0, \pm 2)$; $e = \frac{\sqrt{13}}{2}$; Latus rectum $= 9$
4. Foci $(\pm 10, 0)$, Vertices $(\pm 6, 0)$; $e = \frac{5}{3}$; Latus rectum $= \frac{64}{3}$

5. Foci $(0, \pm\frac{2\sqrt{14}}{\sqrt{5}})$, Vertices $(0, \pm\frac{6}{\sqrt{5}})$; $e = \frac{\sqrt{14}}{3}$; Latus rectum $= \frac{4\sqrt{5}}{3}$

6. Foci $(0, \pm\sqrt{65})$, Vertices $(0, \pm 4)$; $e = \frac{\sqrt{65}}{4}$; Latus rectum $= \frac{49}{2}$

7. $\frac{x^2}{4} - \frac{y^2}{5} = 1$ 8. $\frac{y^2}{25} - \frac{x^2}{39} = 1$ 9. $\frac{y^2}{9} - \frac{x^2}{16} = 1$

10. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ 11. $\frac{y^2}{25} - \frac{x^2}{144} = 1$ 12. $\frac{x^2}{25} - \frac{y^2}{20} = 1$

13. $\frac{x^2}{4} - \frac{y^2}{12} = 1$ 14. $\frac{x^2}{49} - \frac{9y^2}{343} = 1$ 15. $\frac{y^2}{5} - \frac{x^2}{5} = 1$

Miscellaneous Exercise on Chapter 10

1. Focus is at the mid-point of the given diameter.
2. 2.23 m (approx.) 3. 9.11 m (approx.) 4. 1.56m (approx.)
5. $\frac{x^2}{81} + \frac{y^2}{9} = 1$ 6. 18 sq units 7. $\frac{x^2}{25} + \frac{y^2}{9} = 1$
8. $8\sqrt{3}a$

EXERCISE 11.1

1. y and z - coordinates are zero 2. y - coordinate is zero
3. I, IV, VIII, V, VI, II, III, VII
4. (i) XY - plane (ii) $(x, y, 0)$ (iii) Eight

EXERCISE 11.2

1. (i) $2\sqrt{5}$ (ii) $\sqrt{43}$ (iii) $2\sqrt{26}$ (iv) $2\sqrt{5}$
4. $x - 2z = 0$ 5. $9x^2 + 25y^2 + 25z^2 - 225 = 0$

Miscellaneous Exercise on Chapter 11

1. $(1, -2, 8)$ 2. $7, \sqrt{34}, 7$ 3. $a = -2,\ b = -\frac{16}{3},\ c = 2$
4. $x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2 - 109}{2}$

EXERCISE 12.1

1. 6
2. $\left(\pi - \dfrac{22}{7}\right)$
3. π
4. $\dfrac{19}{2}$
5. $-\dfrac{1}{2}$
6. 5
7. $\dfrac{11}{4}$
8. $\dfrac{108}{7}$
9. b
10. 2
11. 1
12. $-\dfrac{1}{4}$
13. $\dfrac{a}{b}$
14. $\dfrac{a}{b}$
15. $\dfrac{1}{\pi}$
16. $\dfrac{1}{\pi}$
17. 4
18. $\dfrac{a+1}{b}$
19. 0
20. 1
21. 0
22. 2
23. 3, 6
24. Limit does not exist at $x = 1$
25. Limit does not exist at $x = 0$
26. Limit does not exist at $x = 0$
27. 0
28. $a = 0, b = 4$
29. $\lim\limits_{x \to a_1} f(x) = 0$ and $\lim\limits_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_n)$
30. $\lim\limits_{x \to a} f(x)$ exists for all $a \neq 0$.
31. 2
32. For $\lim\limits_{x \to 0} f(x)$ to exists, we need $m = n$; $\lim\limits_{x \to 1} f(x)$ exists for any integral value of m and n.

EXERCISE 12.2

1. 20
2. 1
3. 99
4. (i) $3x^2$ (ii) $2x - 3$ (iii) $\dfrac{-2}{x^3}$ (iv) $\dfrac{-2}{(x-1)^2}$
6. $nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1}$
7. (i) $2x - a - b$ (ii) $4ax\left(ax^2 + b\right)$ (iii) $\dfrac{a-b}{(x-b)^2}$

8. $\dfrac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$

9. (i) 2 (ii) $20x^3 - 15x^2 + 6x - 4$ (iii) $\dfrac{-3}{x^4}(5+2x)$ (iv) $15x^4 + \dfrac{24}{x^5}$

(v) $\dfrac{-12}{x^5} + \dfrac{36}{x^{10}}$ (vi) $\dfrac{-2}{(x+1)^2} - \dfrac{x(3x-2)}{(3x-1)^2}$ 10. $-\sin x$

11. (i) $\cos 2x$ (ii) $\sec x \tan x$
(iii) $5\sec x \tan x - 4\sin x$ (iv) $-\operatorname{cosec} x \cot x$
(v) $-3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$ (vi) $5\cos x + 6\sin x$
(vii) $2\sec^2 x - 7\sec x \tan x$

Miscellaneous Exercise on Chapter 12

1. (i) -1 (ii) $\dfrac{1}{x^2}$ (iii) $\cos(x+1)$ (iv) $-\sin\left(x - \dfrac{\pi}{8}\right)$ 2. 1

3. $\dfrac{-qr}{x^2} + ps$ 4. $2c\,(ax+b)(cx+d) + a\,(cx+d)^2$

5. $\dfrac{ad-bc}{(cx+d)^2}$ 6. $\dfrac{-2}{(x-1)^2},\ x \neq 0,1$ 7. $\dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$

8. $\dfrac{-apx^2 - 2bpx + ar - bq}{(px^2+qx+r)^2}$ 9. $\dfrac{apx^2 + 2bpx + bq - ar}{(ax+b)^2}$ 10. $\dfrac{-4a}{x^5} + \dfrac{2b}{x^3} - \sin x$

11. $\dfrac{2}{\sqrt{x}}$ 12. $na(ax+b)^{n-1}$

13. $(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b) + na(cx+d)\right]$ 14. $\cos(x+a)$

15. $-\operatorname{cosec}^3 x - \operatorname{cosec} x \cot^2 x$ 16. $\dfrac{-1}{1+\sin x}$

17. $\dfrac{-2}{(\sin x - \cos x)^2}$ 18. $\dfrac{2\sec x \tan x}{(\sec x + 1)^2}$ 19. $n \sin^{n-1} x \cos x$

20. $\dfrac{bc\cos x + ad\sin x + bd}{(c + d\cos x)^2}$ **21.** $\dfrac{\cos a}{\cos^2 x}$

22. $x^3(5x\cos x + 3x\sin x + 20\sin x - 12\cos x)$

23. $-x^2\sin x - \sin x + 2x\cos x$

24. $-q\,sin x\left(ax^2 + \sin x\right) + (p + q\cos x)(2a\,x + \cos x)$

25. $-\tan^2 x(x + \cos x) + (x - \tan x)(1 - \sin x)$

26. $\dfrac{35 + 15x\cos x + 28\cos x + 28x\sin x - 15\sin x}{(3x + 7\cos x)^2}$

27. $\dfrac{x\cos\frac{\pi}{4}(2\sin x - x\cos x)}{\sin^2 x}$ **28.** $\dfrac{1 + \tan x - x\sec^2 x}{(1 + \tan x)^2}$

29. $(x + \sec x)\left(1 - \sec^2 x\right) + (x - \tan x).\,(1 + \sec x\tan x)$

30. $\dfrac{\sin x - n\,x\cos x}{\sin^{n+1} x}$

EXERCISE 13.1

1. 3 **2.** 8.4 **3.** 2.33 **4.** 7

5. 6.32 **6.** 16 **7.** 3.23 **8.** 5.1

9. 157.92 **10.** 11.28 **11.** 10.34 **12.** 7.35

EXERCISE 13.2

1. 9, 9.25 **2.** $\dfrac{n+1}{2}, \dfrac{n^2-1}{12}$ **3.** 16.5, 74.25 **4.** 19, 43.4

5. 100, 29.09 **6.** 64, 1.69 **7.** 107, 2276 **8.** 27, 132

9. 93, 105.58, 10.27 **10.** 5.55, 43.5

Miscellaneous Exercise on Chapter 13

1. 4, 8 **2.** 6, 8 **3.** 24, 12

5. (i) 10.1, 1.99 (ii) 10.2, 1.98

6. 20, 3.036

EXERCISE 14.1

1. No.

2. (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) ϕ (iii) $\{3, 6\}$ (iv) $\{1, 2, 3\}$ (v) $\{6\}$

(vi) $\{3, 4, 5, 6\}$, $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \phi$, $B \cup C = \{3, 6\}$, $E \cap F = \{6\}$, $D \cap E = \phi$,

$A - C = \{1, 2, 4, 5\}$, $D - E = \{1, 2, 3\}$, $E \cap F' = \phi$, $F' = \{1, 2\}$

3. $A = \{(3,6), (4,5), (5, 4), (6,3), (4,6), (5,5), (6,4), (5,6), (6,5), (6,6)\}$

$B = \{(1,2), (2,2), (3, 2), (4,2), (5,2), (6,2), (2,1), (2,3), (2,4), (2,5), (2,6)\}$

$C = \{(3,6), (6,3), (5, 4), (4,5), (6,6)\}$

A and B, B and C are mutually exclusive.

4. (i) A and B; A and C; B and C; C and D (ii) A and C (iii) B and D

5. (i) "Getting at least two heads", and "getting at least two tails"

(ii) "Getting no heads", "getting exactly one head" and "getting at least two heads"

(iii) "Getting at most two tails", and "getting exactly two tails"

(iv) "Getting exactly one head" and "getting exactly two heads"

(v) "Getting exactly one tail", "getting exactly two tails", and getting exactly three tails"

Note There may be other events also as answer to the above question.

6. $A = \{(2, 1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$B = \{(1, 1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

$C = \{(1, 1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)\}$

(i) $A' = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\} = B$

(ii) $B' = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} = A$

(iii) $A \cup B = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (2,1), (2,2), (2,3), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} = S$

(iv) $A \cap B = \phi$

(v) $A - C = \{(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

(vi) $B \cup C = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

(vii) $B \cap C = \{(1,1), (1,2), (1,3), (1,4), (3,1), (3,2)\}$

(viii) $A \cap B' \cap C' = \{(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

7. (i) True (ii) True (iii) True (iv) False (v) False (vi) False

EXERCISE 14.2

1. (a) Yes (b) Yes (c) No (d) No (e) No   2. $\frac{3}{4}$

3. (i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{6}$ (iv) 0 (v) $\frac{5}{6}$ 4. (a) 52 (b) $\frac{1}{52}$ (c) (i) $\frac{1}{13}$ (ii) $\frac{1}{2}$

5. (i) $\frac{1}{12}$ (ii) $\frac{1}{12}$ 6. $\frac{3}{5}$

7. Rs 4.00 gain, Rs 1.50 gain, Re 1.00 loss, Rs 3.50 loss, Rs 6.00 loss.

P (Winning Rs 4.00) $= \frac{1}{16}$, P(Winning Rs 1.50) $= \frac{1}{4}$, P (Losing Re. 1.00) $= \frac{3}{8}$

P (Losing Rs 3.50) $= \frac{1}{4}$, P (Losing Rs 6.00) $= \frac{1}{16}$.

8. (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$ (iv) $\frac{7}{8}$ (v) $\frac{1}{8}$ (vi) $\frac{1}{8}$ (vii) $\frac{3}{8}$ (viii) $\frac{1}{8}$ (ix) $\frac{7}{8}$

9. $\frac{9}{11}$ 10. (i) $\frac{6}{13}$ (ii) $\frac{7}{13}$ 11. $\frac{1}{38760}$

12. (i) No, because $P(A \cap B)$ must be less than or equal to P(A) and P(B). (ii) Yes

13. (i) $\frac{7}{15}$ (ii) 0.5 (iii) 0.15   14. $\frac{4}{5}$

15. (i) $\frac{5}{8}$ (ii) $\frac{3}{8}$ 16. No   17. (i) 0.58 (ii) 0.52 (iii) 0.74

18. 0.6 **19.** 0.55 **20.** 0.65

21. (i) $\frac{19}{30}$ (ii) $\frac{11}{30}$ (iii) $\frac{2}{15}$

Miscellaneous Exercise on Chapter 14

1. (i) $\frac{^{20}C_5}{^{60}C_5}$ (ii) $1-\frac{^{30}C_5}{^{60}C_5}$ **2.** $\frac{^{13}C_3 \cdot {}^{13}C_1}{^{52}C_4}$

3. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{5}{6}$ **4.** (a) $\frac{999}{1000}$ (b) $\frac{^{9990}C_2}{^{10000}C_2}$ (c) $\frac{^{9990}C_{10}}{^{10000}C_{10}}$

5. (a) $\frac{17}{33}$ (b) $\frac{16}{33}$ **6.** $\frac{2}{3}$

7. (i) 0.88 (ii) 0.12 (iii) 0.19 (iv) 0.34 **8.** $\frac{4}{5}$

9. (i) $\frac{33}{83}$ (ii) $\frac{3}{8}$ **10.** $\frac{1}{5040}$

SUPPLEMENTARY MATERIAL

CHAPTER 8

8.6 Infinite G.P. and its Sum

G.P. of the form $a, ar, ar^2, ar^3, \ldots$ is called infinite G.P. Now, to find the formulae for finding sum to infinity of a G.P., we begin with an example.

Let us consider the G.P.,

$$1, \frac{2}{3}, \frac{4}{9}, \ldots$$

Here $a = 1$, $r = \frac{2}{3}$. We have

$$S_n = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger:

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, $\left(\frac{2}{3}\right)^n$ becomes sufficiently small. In other words as $n \to \infty, \left(\frac{2}{3}\right)^n \to 0$. Consequently, we find that the sum of infinitely many terms is given by $S_\infty = 3$.

Now, for a geometric progression, a, ar, ar^2, ..., if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case as $n \to \infty$, $r^n \to 0$ since $|r| < 1$. Therefore

$$S_n \to \frac{a}{1-r}$$

Symbolically sum to infinity is denoted by S_∞ or S.

Thus, we have $S = \frac{a}{1-r}$.

For examples,

(i) $$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1}{1-\frac{1}{2}} = 2.$$

(ii) $$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots = \frac{1}{1-\left(\frac{-1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

Exercise 8.3

Find the sum to infinity in each of the following Geometric Progression.

1. $1, \frac{1}{3}, \frac{1}{9}, \ldots$ (Ans. 1.5)
2. 6, 1.2, .24, ... (Ans. 7.5)
3. $5, \frac{20}{7}, \frac{80}{49}, \ldots$ (Ans. $\frac{35}{3}$)
4. $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, \ldots$ (Ans. $\frac{-3}{5}$)
5. Prove that $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \ldots = 3$
6. Let $x = 1 + a + a^2 + \ldots$ and $y = 1 + b + b^2 + \ldots$, where $|a| < 1$ and $|b| < 1$. Prove that

$$1 + ab + a^2b^2 + \ldots = \frac{xy}{x+y-1}$$

CHAPTER 12

12.6 Limits Involving Exponential and Logarithmic Functions

Before discussing evaluation of limits of the expressions involving exponential and logarithmic functions, we introduce these two functions stating their domain, range and also sketch their graphs roughly.

Leonhard Euler (1707–1783), the great Swiss mathematician introduced the number e whose value lies between 2 and 3. This number is useful in defining exponential function and is defined as $f(x) = e^x$, $x \in \mathbf{R}$. Its domain is $\mathbf{R}$, range is the set of positive real numbers. The graph of exponential function, i.e., $y = e^x$ is as given in Fig.13.11.

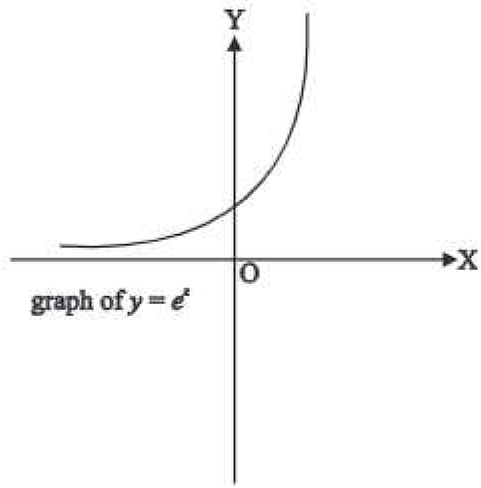


Fig. 13.11

Similarly, the logarithmic function expressed as $\log_e \mathbf{R}^+ \to \mathbf{R}$ is given by $\log_e x = y$, if and only if $e^y = x$. Its domain is $\mathbf{R}^+$ which is the set of all positive real numbers and range is $\mathbf{R}$. The graph of logarithmic function $y = \log_e x$ is shown in Fig.13.12.

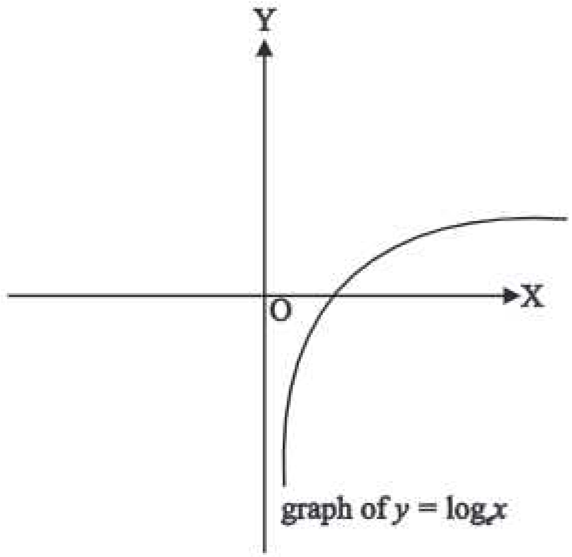


Fig. 13.12

In order to prove the result $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$, we make use of an inequality involving the expression $\frac{e^x - 1}{x}$ which runs as follows:

$\frac{1}{1+|x|} \le \frac{e^x - 1}{x} \le 1 + (e-2)|x|$ holds for all x in $[-1, 1] \sim \{0\}$.

Theorem 6 Prove that $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof Using above inequality, we get

$$\frac{1}{1+|x|} \le \frac{e^x - 1}{x} \le 1 + |x|(e-2),\ x \hat{I} [-1, 1] \sim \{0\}$$

Also $\quad \lim_{x \to 0} \frac{1}{1+|x|} = \frac{1}{1 + \lim_{x \to 0} |x|} = \frac{1}{1+0} = 1$

and $\lim_{x \to 0} \left[1 + (e-2)|x|\right] = 1 + (e-2)\lim_{x \to 0}|x| = 1 + (e-2)0 = 1$

Therefore, by Sandwich theorem, we get

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Theorem 7 Prove that $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$

Proof Let $\frac{\log_e(1+x)}{x} = y$. Then

$$\log_e(1+x) = xy$$

$$\Rightarrow 1 + x = e^{xy}$$

$$\Rightarrow \frac{e^{xy} - 1}{x} = 1$$

or

$$\frac{e^{xy} - 1}{xy} . y = 1$$

$$\Rightarrow \lim_{xy \to 0} \frac{e^{xy} - 1}{xy} \lim_{x \to 0} y = 1 \text{ (since } x \to 0 \text{ gives } xy \to 0)$$

$$\Rightarrow \lim_{x \to 0} y = 1 \left(\text{as} \lim_{xy \to 0} \frac{e^{xy} - 1}{xy} = 1 \right)$$

$$\Rightarrow \lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$$

Example 5 Compute $\lim_{x \to 0} \frac{e^{3x} - 1}{x}$

Solution We have

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x} = \lim_{3x \to 0} \frac{e^{3x} - 1}{3x} \cdot 3$$

$$= 3\left(\lim_{y \to 0} \frac{e^y - 1}{y} \right), \quad \text{where } y = 3x$$

$$= 3.1 = 3$$

Example 6 Compute $\lim_{x \to 0} \frac{e^x - \sin x - 1}{x}$

Solution We have $\lim_{x \to 0} \frac{e^x - \sin x - 1}{x} = \lim_{x \to 0} \left[\frac{e^x - 1}{x} - \frac{\sin x}{x} \right]$

$$= \lim_{x \to 0} \frac{e^x - 1}{x} - \lim_{x \to 0} \frac{\sin x}{x} = 1 - 1 = 0$$

Example 7 Evaluate $\lim_{x \to 1} \frac{\log_e x}{x - 1}$

Solution Put $x = 1 + h$, then as $x \to 1 \Rightarrow h \to 0$. Therefore,

$$\lim_{x \to 1} \frac{\log_e x}{x - 1} = \lim_{h \to 0} \frac{\log_e (1 + h)}{h} = 1 \left(\text{since } \lim_{x \to 0} \frac{\log_e (1 + x)}{x} = 1 \right).$$

Exercise 13.2

Evaluate the following limits, if exist

1. $\lim_{x \to 0} \frac{e^{4x} - 1}{x}$ (Ans. 4)

2. $\lim_{x \to 0} \frac{e^{2+x} - e^2}{x}$ (Ans. e^2)

3. $\lim_{x \to 5} \frac{e^x - e^5}{x - 5}$ (Ans. e^5)

4. $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}$ (Ans. 1)

5. $\lim_{x \to 3} \frac{e^x - e^3}{x - 3}$ (Ans. e^3)

6. $\lim_{x \to 0} \frac{x(e^x - 1)}{1 - \cos x}$ (Ans. 2)

7. $\lim_{x \to 0} \frac{\log_e (1 + 2x)}{x}$ (Ans. 2)

8. $\lim_{x \to 0} \frac{\log (1 + x^3)}{\sin^3 x}$ (Ans. 1)

$$= \lim_{h \to 0} \frac{\sin h(\cos x - \sin x) + \sin x(\cos h - 1) + \cos x(\cos h - 1)}{h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h}(\cos x - \sin x) + \lim_{h \to 0} \sin x \frac{(\cos h - 1)}{h} + \lim_{h \to 0} \cos x \frac{(\cos h - 1)}{h}$$

$$= \cos x - \sin x$$

(ii) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)\sin(x+h) - x\sin x}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(\sin x \cos h + \sin h \cos x) - x\sin x}{h}$$

$$= \lim_{h \to 0} \frac{x\sin x(\cos h - 1) + x\cos x \sin h + h(\sin x \cos h + \sin h \cos x)}{h}$$

$$= \lim_{h \to 0} \frac{x\sin x(\cos h - 1)}{h} + \lim_{h \to 0} x\cos x \frac{\sin h}{h} + \lim_{h \to 0}(\sin x \cos h + \sin h \cos x)$$

$$= x\cos x + \sin x$$

Example 21 Compute derivative of

(i) $f(x) = \sin 2x$ (ii) $g(x) = \cot x$

Solution (i) Recall the trigonometric formula $\sin 2x = 2 \sin x \cos x$. Thus

$$\frac{df(x)}{dx} = \frac{d}{dx}(2\sin x \cos x) = 2\frac{d}{dx}(\sin x \cos x)$$

$$= 2\left[(\sin x)' \cos x + \sin x (\cos x)'\right]$$

$$= 2\left[(\cos x)\cos x + \sin x(-\sin x)\right]$$

$$= 2(\cos^2 x - \sin^2 x)$$

(ii) By definition, $g(x) = \cot x = \frac{\cos x}{\sin x}$. We use the quotient rule on this function wherever it is defined. $\frac{dg}{dx} = \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

$$= \frac{(\cos x)'(\sin x) - (\cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Alternatively, this may be computed by noting that $\cot x = \frac{1}{\tan x}$. Here, we use the fact that the derivative of *tan x* is *sec² x* which we saw in Example 17 and also that the derivative of the constant function is 0.

$$\frac{dg}{dx} = \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{(1)'(\tan x) - (1)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(0)(\tan x) - (\sec x)^2}{(\tan x)^2}$$

$$= \frac{-\sec^2 x}{\tan^2 x} = -\operatorname{cosec}^2 x$$

Example 22 Find the derivative of

(i) $\frac{x^5 - \cos x}{\sin x}$ (ii) $\frac{x + \cos x}{\tan x}$

Solution (i) Let $h(x) = \frac{x^5 - \cos x}{\sin x}$. We use the quotient rule on this function wherever it is defined.

$$h'(x) = \frac{(x^5 - \cos x)'\sin x - (x^5 - \cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(5x^4 + \sin x)\sin x - (x^5 - \cos x)\cos x}{\sin^2 x}$$

$$= \frac{-x^5 \cos x + 5x^4 \sin x + 1}{(\sin x)^2}$$

(ii) We use quotient rule on the function $\frac{x + \cos x}{\tan x}$ wherever it is defined.

$$h'(x) = \frac{(x + \cos x)' \tan x - (x + \cos x)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(1 - \sin x)\tan x - (x + \cos x)\sec^2 x}{(\tan x)^2}$$

Miscellaneous Exercise on Chapter 12

1. Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin(x + 1)$ (iv) $\cos\left(x - \frac{\pi}{8}\right)$

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

2. $(x + a)$

3. $(px + q)\left(\frac{r}{x} + s\right)$

4. $(ax + b)(cx + d)^2$

5. $\frac{ax + b}{cx + d}$

6. $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$

7. $\frac{1}{ax^2 + bx + c}$

8. $\frac{ax + b}{px^2 + qx + r}$

9. $\frac{px^2 + qx + r}{ax + b}$

10. $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

11. $4\sqrt{x} - 2$

12. $(ax + b)^n$

13. $(ax + b)^n (cx + d)^m$

14. $\sin(x + a)$

15. $\operatorname{cosec} x \cot x$

16. $\frac{\cos x}{1 + \sin x}$

17. $\dfrac{\sin x+\cos x}{\sin x-\cos x}$ 18. $\dfrac{\sec x-1}{\sec x+1}$ 19. $\sin^n x$

20. $\dfrac{a+b\sin x}{c+d\cos x}$ 21. $\dfrac{\sin(x+a)}{\cos x}$ 22. $x^4(5\sin x-3\cos x)$

23. $\left(x^2+1\right)\cos x$ 24. $\left(ax^2+\sin x\right)\left(p+q\cos x\right)$

25. $\left(x+\cos x\right)\left(x-\tan x\right)$ 26. $\dfrac{4x+5\sin x}{3x+7\cos x}$ 27. $\dfrac{x^2\cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

28. $\dfrac{x}{1+\tan x}$ 29. $\left(x+\sec x\right)\left(x-\tan x\right)$ 30. $\dfrac{x}{\sin^n x}$

Summary

- The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point. Similarly the right hand limit.
- Limit of a function at a point is the common value of the left and right hand limits, if they coincide.
- For a function f and a real number a, $\lim_{x\to a} f(x)$ and $f(a)$ may not be same (In fact, one may be defined and not the other one).
- For functions f and g the following holds:

$$\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x)$$

$$\lim_{x\to a}\left[f(x).g(x)\right]=\lim_{x\to a}f(x).\lim_{x\to a}g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$$

- Following are some of the standard limits

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

- The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Derivative of a function f at any point x is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- For functions u and v the following holds:

$$(u \pm v)' = u' \pm v'$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$ provided all are defined.

- Following are some of the standard derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Historical Note

In the history of mathematics two names are prominent to share the credit for inventing calculus, Issac Newton (1642 – 1727) and G.W. Leibnitz (1646 – 1717). Both of them independently invented calculus around the seventeenth century. After the advent of calculus many mathematicians contributed for further development of calculus. The rigorous concept is mainly attributed to the great mathematicians, A.L. Cauchy, J.L.Lagrange and Karl Weierstrass. Cauchy gave the foundation of calculus as we have now generally accepted in our textbooks. Cauchy used D' Alembert's limit concept to define the derivative of a function. Starting with definition of a limit, Cauchy gave examples such as the limit of $\frac{\sin \alpha}{\alpha}$ for $\alpha = 0$. He wrote $\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}$, and called the limit for $i \to 0$, the "function derive'e, y' for $f'(x)$".

Before 1900, it was thought that calculus is quite difficult to teach. So calculus became beyond the reach of youngsters. But just in 1900, John Perry and others in England started propagating the view that essential ideas and methods of calculus were simple and could be taught even in schools. F.L. Griffin, pioneered the teaching of calculus to first year students. This was regarded as one of the most daring act in those days.

Today not only the mathematics but many other subjects such as Physics, Chemistry, Economics and Biological Sciences are enjoying the fruits of calculus.

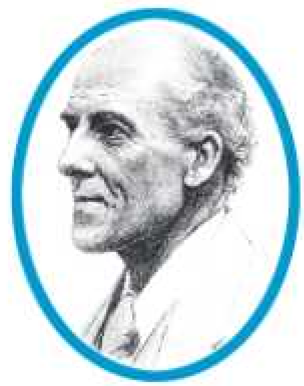
— ❖ —

Chapter 13

STATISTICS

❖ *"Statistics may be rightly called the science of averages and their estimates." – A.L.BOWLEY & A.L. BODDINGTON* ❖

13.1 Introduction



Karl Pearson (1857-1936)

We know that statistics deals with data collected for specific purposes. We can make decisions about the data by analysing and interpreting it. In earlier classes, we have studied methods of representing data graphically and in tabular form. This representation reveals certain salient features or characteristics of the data. We have also studied the methods of finding a representative value for the given data. This value is called the measure of central tendency. Recall mean (arithmetic mean), median and mode are three measures of central tendency. A *measure of central tendency* gives us a rough idea where data points are centred. But, in order to make better interpretation from the data, we should also have an idea how the data are scattered or how much they are bunched around a measure of central tendency.

Consider now the runs scored by two batsmen in their last ten matches as follows:

Batsman A : 30, 91, 0, 64, 42, 80, 30, 5, 117, 71

Batsman B : 53, 46, 48, 50, 53, 53, 58, 60, 57, 52

Clearly, the mean and median of the data are

	Batsman A	Batsman B
Mean	53	53
Median	53	53

Recall that, we calculate the mean of a data (denoted by $\bar{x}$) by dividing the sum of the observations by the number of observations, i.e.,

Therefore, the coordinates of the foci are (0, $\pm\sqrt{17}$) and that of the vertices are (0, ± 4). Also,

The eccentricity $e=\frac{c}{a}=\frac{\sqrt{17}}{4}$. The latus rectum $=\frac{2b^2}{a}=\frac{1}{2}$.

Example 15 Find the equation of the hyperbola with foci (0, ± 3) and vertices (0, $\pm\frac{\sqrt{11}}{2}$).

Solution Since the foci is on y-axis, the equation of the hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

Since vertices are (0, $\pm\frac{\sqrt{11}}{2}$), $a=\frac{\sqrt{11}}{2}$

Also, since foci are (0, ± 3); $c = 3$ and $b^2 = c^2 - a^2 = \frac{25}{4}$.

Therefore, the equation of the hyperbola is

$$\frac{y^2}{\left(\frac{11}{4}\right)}-\frac{x^2}{\left(\frac{25}{4}\right)}=1, \text{ i.e., } 100\,y^2 - 44\,x^2 = 275.$$

Example 16 Find the equation of the hyperbola where foci are (0, ±12) and the length of the latus rectum is 36.

Solution Since foci are (0, ± 12), it follows that $c = 12$.

Length of the latus rectum $=\frac{2b^2}{a}=36$ or $b^2 = 18a$

Therefore $\quad c^2 = a^2 + b^2$; gives

$144 = a^2 + 18a$

i.e., $\quad a^2 + 18a - 144 = 0$,

So $\quad a = -24, 6$.

Since a cannot be negative, we take $a = 6$ and so $b^2 = 108$.

Therefore, the equation of the required hyperbola is $\frac{y^2}{36}-\frac{x^2}{108}=1$, i.e., $3y^2 - x^2 = 108$

EXERCISE 10.4

In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas.

1. $\frac{x^2}{16}-\frac{y^2}{9}=1$
2. $\frac{y^2}{9}-\frac{x^2}{27}=1$
3. $9y^2 - 4x^2 = 36$
4. $16x^2 - 9y^2 = 576$
5. $5y^2 - 9x^2 = 36$
6. $49y^2 - 16x^2 = 784$.

In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions.

7. Vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$
8. Vertices $(0, \pm 5)$, foci $(0, \pm 8)$
9. Vertices $(0, \pm 3)$, foci $(0, \pm 5)$
10. Foci $(\pm 5, 0)$, the transverse axis is of length 8.
11. Foci $(0, \pm 13)$, the conjugate axis is of length 24.
12. Foci $(\pm 3\sqrt{5}, 0)$, the latus rectum is of length 8.
13. Foci $(\pm 4, 0)$, the latus rectum is of length 12
14. vertices $(\pm 7, 0)$, $e = \frac{4}{3}$.
15. Foci $(0, \pm \sqrt{10})$, passing through (2,3)

Miscellaneous Examples

Example 17 The focus of a parabolic mirror as shown in Fig 10.31 is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB (Fig 10.31).

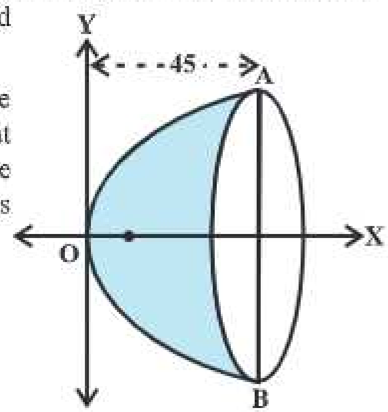


Fig 10.31

Solution Since the distance from the focus to the vertex is 5 cm. We have, $a = 5$. If the origin is taken at the vertex and the axis of the mirror lies along the positive x-axis, the equation of the parabolic section is

$$y^2 = 4\,(5)\,x = 20\,x$$

Note that $x = 45$. Thus

$$y^2 = 900$$

Therefore $y = \pm 30$

Hence $AB = 2y = 2 \times 30 = 60$ cm.

Example 18 A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?

Solution Let the vertex be at the lowest point and the axis vertical. Let the coordinate axis be chosen as shown in Fig 10.32.

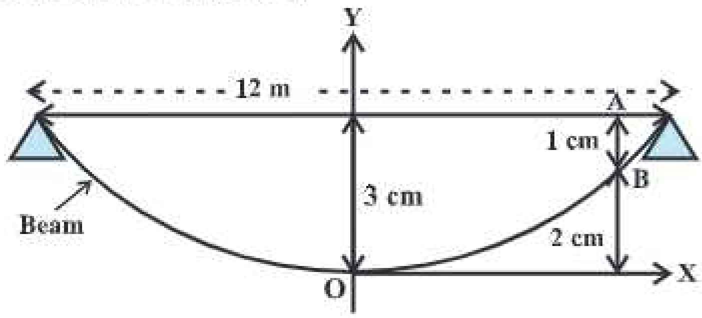


Fig 10.32

The equation of the parabola takes the form $x^2 = 4ay$. Since it passes through $\left(6, \frac{3}{100}\right)$, we have $(6)^2 = 4a\left(\frac{3}{100}\right)$, i.e., $a = \frac{36 \times 100}{12} = 300$ m

Let AB be the deflection of the beam which is $\frac{1}{100}$ m. Coordinates of B are $\left(x, \frac{2}{100}\right)$.

Therefore $\qquad x^2 = 4 \times 300 \times \frac{2}{100} = 24$

i.e. $\qquad x = \sqrt{24} = 2\sqrt{6}$ metres

Example 19 A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point P(x, y) is taken on the rod in such a way that AP = 6 cm. Show that the locus of P is an ellipse.

Solution Let AB be the rod making an angle θ with OX as shown in Fig 10.33 and P (x, y) the point on it such that AP = 6 cm.

Since $\qquad$ AB = 15 cm, we have

$\qquad$ PB = 9 cm.

From P draw PQ and PR perpendiculars on y-axis and x-axis, respectively.

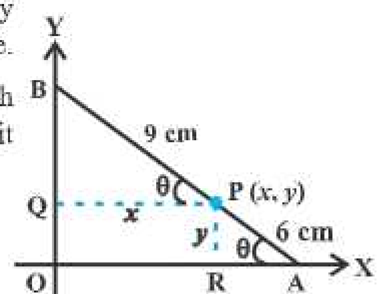


Fig 10.33

From Δ PBQ, $\cos\theta = \frac{x}{9}$

From Δ PRA, $\sin\theta = \frac{y}{6}$

Since $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

or $$\frac{x^2}{81} + \frac{y^2}{36} = 1$$

Thus the locus of P is an ellipse.

Miscellaneous Exercise on Chapter 10

1. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.
2. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?
3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.
4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.
5. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the *x*-axis.
6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.
7. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.
8. An equilateral triangle is inscribed in the parabola $y^2 = 4\,ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Summary

In this Chapter the following concepts and generalisations are studied.

- A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.
- The equation of a circle with centre (h, k) and the radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

- A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane.
- The equation of the parabola with focus at $(a, 0)$ $a > 0$ and directrix $x = -a$ is

$$y^2 = 4ax.$$

- Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola.
- Length of the latus rectum of the parabola $y^2 = 4ax$ is $4a$.
- An *ellipse* is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.
- The equation of an ellipse with foci on the x-axis is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
- Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse.
- Length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}$.
- The eccentricity of an ellipse is the ratio between the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse.
- A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.
- The equation of a hyperbola with foci on the x-axis is : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

- Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.
- Length of the latus rectum of the hyperbola : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is : $\frac{2b^2}{a}$.
- The eccentricity of a hyperbola is the ratio of the distances from the centre of the hyperbola to one of the foci and to one of the vertices of the hyperbola.

Historical Note

Geometry is one of the most ancient branches of mathematics. The Greek geometers investigated the properties of many curves that have theoretical and practical importance. Euclid wrote his treatise on geometry around 300 B.C. He was the first who organised the geometric figures based on certain axioms suggested by physical considerations. Geometry as initially studied by the ancient Indians and Greeks, who made essentially no use of the process of algebra. The synthetic approach to the subject of geometry as given by Euclid and in *Sulbasutras*, etc., was continued for some 1300 years. In the 200 B.C., Apollonius wrote a book called '*The Conic*' which was all about conic sections with many important discoveries that have remained unsurpassed for eighteen centuries.

Modern analytic geometry is called '*Cartesian*' after the name of Rene Descartes (1596-1650) whose relevant 'La Geometrie' was published in 1637. But the fundamental principle and method of analytical geometry were already discovered by Pierre de Fermat (1601-1665). Unfortunately, Fermats treatise on the subject, entitled *Ad Locus Planos et So LIDOS Isagoge* (Introduction to Plane and Solid Loci) was published only posthumously in 1679. So, Descartes came to be regarded as the unique inventor of the analytical geometry.

Isaac Barrow avoided using cartesian method. Newton used method of undetermined coefficients to find equations of curves. He used several types of coordinates including polar and bipolar. Leibnitz used the terms '*abscissa*', 'ordinate' and 'coordinate'. L' Hospital (about 1700) wrote an important textbook on analytical geometry.

Clairaut (1729) was the first to give the distance formula although in clumsy form. He also gave the intercept form of the linear equation. Cramer (1750) made formal use of the two axes and gave the equation of a circle as

$$(y - a)^2 + (b - x)^2 = r$$

He gave the best exposition of the analytical geometry of his time. Monge (1781) gave the modern 'point-slope' form of equation of a line as

$$y - y' = a\,(x - x')$$

and the condition of perpendicularity of two lines as $aa' + 1 = 0$.

S.F. Lacroix (1765–1843) was a prolific textbook writer, but his contributions to analytical geometry are found scattered. He gave the 'two-point' form of equation of a line as

$$y - \beta = \frac{\beta' - \beta}{\alpha' - \alpha}(x - \alpha)$$

and the length of the perpendicular from (α, β) on $y = ax + b$ as $\dfrac{(\beta - a - b)}{\sqrt{1 + a^2}}$.

His formula for finding angle between two lines was $\tan 0 = \left(\dfrac{a' - a}{1 + aa'}\right)$. It is, of course, surprising that one has to wait for more than 150 years after the invention of analytical geometry before finding such essential basic formula. In 1818, C. Lame, a civil engineer, gave $mE + m'E' = 0$ as the curve passing through the points of intersection of two loci $E = 0$ and $E' = 0$.

Many important discoveries, both in Mathematics and Science, have been linked to the conic sections. The Greeks particularly Archimedes (287–212 B.C.) and Apollonius (200 B.C.) studied conic sections for their own beauty. These curves are important tools for present day exploration of outer space and also for research into behaviour of atomic particles.

Chapter 11

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

❖*Mathematics is both the queen and the hand-maiden of all sciences – E.T. BELL*❖

11.1 Introduction

Leonhard Euler (1707-1783)

You may recall that to locate the position of a point in a plane, we need two intersecting mutually perpendicular lines in the plane. These lines are called the *coordinate axes* and the two numbers are called the *coordinates of the point with respect to the axes*. In actual life, we do not have to deal with points lying in a plane only. For example, consider the position of a ball thrown in space at different points of time or the position of an aeroplane as it flies from one place to another at different times during its flight.

Similarly, if we were to locate the position of the lowest tip of an electric bulb hanging from the ceiling of a room or the position of the central tip of the ceiling fan in a room, we will not only require the perpendicular distances of the point to be located from two perpendicular walls of the room but also the height of the point from the floor of the room. Therefore, we need not only two but three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor of the room and two adjacent walls of the room. The three numbers representing the three distances are called the *coordinates of the point with reference to the three coordinate planes*. So, a point in space has three coordinates. In this Chapter, we shall study the basic concepts of geometry in three dimensional space.*

* For various activities in three dimensional geometry one may refer to the Book, "***A Hand Book for designing Mathematics Laboratory in Schools***", NCERT, 2005.

Chapter 9

STRAIGHT LINES

❖ *Geometry, as a logical system, is a means and even the most powerful means to make children feel the strength of the human spirit that is of their own spirit.* – *H. FREUDENTHAL* ❖

9.1 Introduction

We are familiar with two-dimensional *coordinate geometry* from earlier classes. Mainly, it is a combination of *algebra* and *geometry*. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes, in his book 'La Géométry. published in 1637. This book introduced the notion of the equation of a curve and related analytical methods into the study of geometry. The resulting combination of analysis and geometry is referred now as *analytical geometry*. In the earlier classes, we initiated the study of coordinate geometry, where we studied about coordinate axes, coordinate plane, plotting of points in a plane, distance between two points, section formulae, etc. All these concepts are the basics of coordinate geometry.

René Descartes (1596 -1650)

Let us have a brief recall of coordinate geometry done in earlier classes. To recapitulate, the location of the points (6, – 4) and (3, 0) in the XY-plane is shown in Fig 9.1.

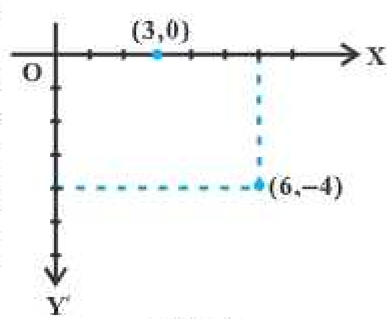


Fig 9.1

We may note that the point (6, – 4) is at 6 units distance from the y-axis measured along the positive x-axis and at 4 units distance from the x-axis measured along the negative y-axis. Similarly, the point (3, 0) is at 3 units distance from the y-axis measured along the positive x-axis and has zero distance from the x-axis.

We also studied there following important formulae:

Let us name the three pants as P_1, P_2, P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the Fig. 6.1.

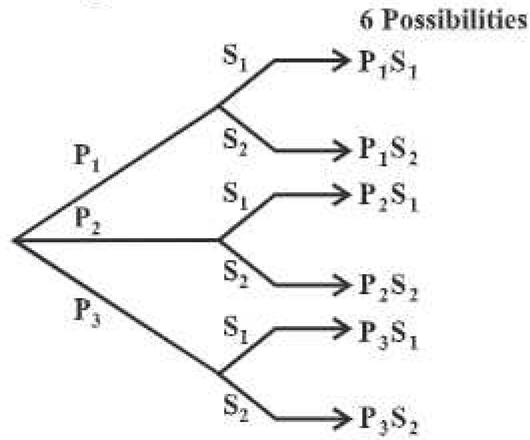


Fig 6.1

Let us consider another problem of the same type.

Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are $6 \times 2 = 12$ different ways in which, Sabnam can carry these items to school. If we name the 2 school bags as B_1, B_2, the three tiffin boxes as T_1, T_2, T_3 and the two water bottles as W_1, W_2, these possibilities can be illustrated in the Fig. 6.2.

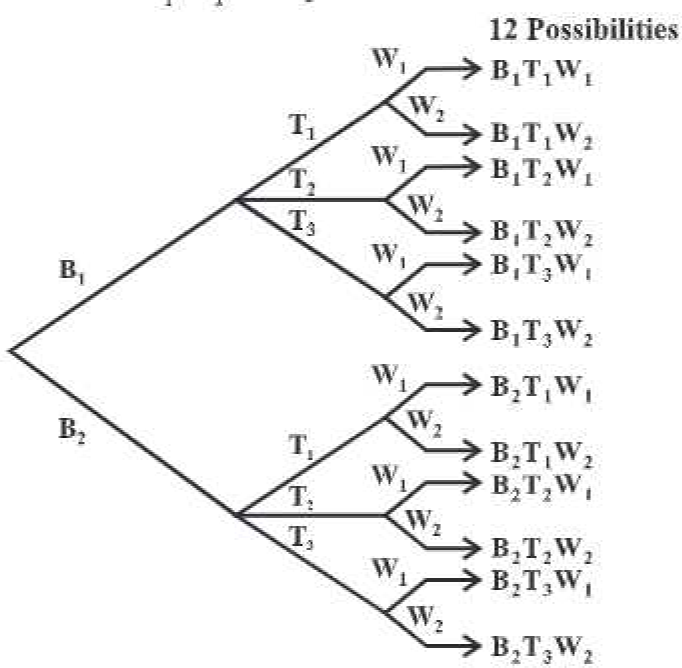


Fig 6.2

In fact, the problems of the above types are solved by applying the following principle known as the *fundamental principle of counting*, or, simply, the *multiplication principle*, which states that

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$*."*

The above principle can be generalised for any finite number of events. For example, for 3 events, the principle is as follows:

'If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to 'the events in the given order is $m \times n \times p$."

In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a pant
(ii) the event of choosing a shirt.

In the second problem, the required number of ways was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a school bag
(ii) the event of choosing a tiffin box
(iii) the event of choosing a water bottle.

Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurence of the events in this chosen order.

Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in 4 vacant places □ □ □ □ by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R,O,S,E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

Note If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $4 \times 4 \times 4 \times 4 = 256$.

Example 2 Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = $4 \times 3 = 12$.

Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

Example 4 Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 flags.

The number of ways is $5 \times 4 \times 3 = 60$.

Continuing the same way, we find that

The number of 4 flag signals $= 5 \times 4 \times 3 \times 2 = 120$

and the number of 5 flag signals $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals $= 20 + 60 + 120 + 120 = 320$.

EXERCISE 6.1

1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that
   (i) repetition of the digits is allowed?
   (ii) repetition of the digits is not allowed?
2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?
3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?
4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?
5. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?
6. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

6.3 Permutations

In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a *permutation of* 4 *different letters taken all at a time*. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words $= 6 \times 5 \times 4 = 120$ (by using multiplication principle).

If the repetition of the letters was allowed, the required number of words would be $6 \times 6 \times 6 = 216$.

Definition 1 A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

In the following sub-section, we shall obtain the formula needed to answer these questions immediately.

6.3.1 *Permutations when all the objects are distinct*

Theorem 1 The number of permutations of n different objects taken r at a time, where $0 < r \le n$ and the objects do not repeat is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by ${}^nP_r$.

Proof There will be as many permutations as there are ways of filling in r vacant places $\square\ \square\ \square \ldots \square$ by

$\leftarrow r$ vacant places $\rightarrow$

the n objects. The first place can be filled in n ways; following which, the second place can be filled in $(n-1)$ ways, following which the third place can be filled in $(n-2)$ ways,..., the rth place can be filled in $(n-(r-1))$ ways. Therefore, the number of ways of filling in r vacant places in succession is $n(n-1)(n-2)\ldots(n-(r-1))$ or $n(n-1)(n-2)\ldots(n-r+1)$

This expression for ${}^nP_r$ is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol $n!$ (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually $n!$ means.

6.3.2 ***Factorial notation*** The notation $n!$ represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as $n!$. We read this symbol as 'n factorial'. Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n!$

$$1 = 1!$$
$$1 \times 2 = 2!$$
$$1 \times 2 \times 3 = 3!$$
$$1 \times 2 \times 3 \times 4 = 4! \text{ and so on.}$$

We define $0! = 1$

We can write $5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2!$
$$= 5 \times 4 \times 3 \times 2 \times 1!$$

Clearly, for a natural number n

$$n! = n(n-1)!$$
$$= n(n-1)(n-2)! \qquad [\text{provided } (n \ge 2)]$$
$$= n(n-1)(n-2)(n-3)! \qquad [\text{provided } (n \ge 3)]$$

and so on.

Example 5 Evaluate (i) 5 ! (ii) 7 ! (iii) 7 ! – 5!

Solution (i) $5\,! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7\,! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

and (iii) $7\,! - 5! = 5040 - 120 = 4920$.

Example 6 Compute (i) $\frac{7!}{5!}$ (ii) $\frac{12!}{(10!)\,(2!)}$

Solution (i) We have $\frac{7!}{5!} = \frac{7\times 6\times 5!}{5!} = 7 \times 6 = 42$

and (ii) $\frac{12!}{(10!)\,(2!)} = \frac{12\times 11\times (10!)}{(10!)\times (2)} = 6 \times 11 = 66$.

Example 7 Evaluate $\frac{n!}{r!(n-r)!}$, when $n = 5$, $r = 2$.

Solution We have to evaluate $\frac{5!}{2!(5-2)!}$ (since $n = 5$, $r = 2$)

We have $\frac{5!}{2\,!(5-2)!} = \frac{5!}{2!\times 3!} = \frac{5\times 4}{2} = 10$.

Example 8 If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, find x.

Solution We have $\frac{1}{8!} + \frac{1}{9\times 8!} = \frac{x}{10\times 9\times 8!}$

Therefore $1+\frac{1}{9} = \frac{x}{10\times 9}$ or $\frac{10}{9} = \frac{x}{10\times 9}$

So $x = 100$.

EXERCISE 6.2

1. Evaluate

(i) 8 ! (ii) 4 ! – 3 !

2. Is 3 ! + 4 ! = 7 ! ?

3. Compute $\frac{8!}{6! \times 2!}$

4. If $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$, find x

5. Evaluate $\frac{n!}{(n-r)!}$, when

(i) $n = 6, r = 2$ (ii) $n = 9, r = 5$.

6.3.3 *Derivation of the formula for* ^nP_r

$$^nP_r = \frac{n!}{(n-r)!},\ 0 \le r \le n$$

Let us now go back to the stage where we had determined the following formula:

$$^nP_r = n\,(n-1)\,(n-2)\,.\,.\,.\,(n-r+1)$$

Multiplying numerator and denomirator by $(n-r)\,(n-r-1)\,.\,.\,.\,3 \times 2 \times 1$, we get

$$^nP_r = \frac{n(n-1)(n-2)...(n-r+1)(n-r)(n-r-1)...3\times2\times1}{(n-r)(n-r-1)...3\times2\times1} = \frac{n!}{(n-r)!},$$

Thus $^nP_r = \frac{n!}{(n-r)!}$, where $0 < r \le n$

This is a much more convenient expression for ^nP_r than the previous one.

In particular, when $r = n$, $^nP_n = \frac{n!}{0!} = n!$

Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we know that there is only one way of doing so. Thus, we can have

$$^nP_0 = 1 = \frac{n!}{n!} = \frac{n!}{(n-0)!} \qquad ... (1)$$

Therefore, the formula (1) is applicable for $r = 0$ also.

Thus $^nP_r = \frac{n!}{(n-r)!},\ 0 \le r \le n$.

Theorem 2 The number of permutations of n different objects taken r at a time, where repetition is allowed, is n^r.

Proof is very similar to that of Theorem 1 and is left for the reader to arrive at.

Here, we are solving some of the problems of the pervious Section using the formula for ${}^nP_r$ to illustrate its usefulness.

In Example 1, the required number of words = ${}^4P_4 = 4! = 24$. Here repetition is not allowed. If repetition is allowed, the required number of words would be $4^4 = 256$.

The number of 3-letter words which can be formed by the letters of the word NUMBER = ${}^6P_3 = \frac{6!}{3!} = 4 \times 5 \times 6 = 120$. Here, in this case also, the repetition is not allowed. If the repetition is allowed,the required number of words would be $6^3 = 216$.

The number of ways in which a Chairman and a Vice-Chairman can be chosen from amongst a group of 12 persons assuming that one person can not hold more than one position, clearly ${}^{12}P_2 = \frac{12!}{10!} = 11 \times 12 = 132$.

6.3.4 ***Permutations when all the objects are not distinct objects*** Suppose we have to find the number of ways of rearranging the letters of the word ROOT. In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind. Let us treat, temporarily, the 2 Os as different, say, O_1 and O_2. The number of permutations of 4-different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, RO_1O_2T. Corresponding to this permutation,we have 2 ! permutations RO_1O_2T and RO_2O_1T which will be exactly the same permutation if O_1 and O_2 are not treated as different, i.e., if O_1 and O_2 are the same O at both places.

Therefore, the required number of permutations = $\frac{4!}{2!} = 3 \times 4 = 12$.

Permutations when O_1, O_2 are different.		Permutations when O_1, O_2 are the same O.
$\left.\begin{matrix} RO_1O_2T \\ RO_2O_1T \end{matrix}\right]$	$\longrightarrow$	R O O T
$\left.\begin{matrix} TO_1O_2R \\ TO_2O_1R \end{matrix}\right]$	$\longrightarrow$	T O O R

$R O_1 T O_2$ $R O_2 T O_1$	$\longrightarrow$	R O T O
$T O_1 R O_2$ $T O_2 R O_1$	$\longrightarrow$	T O R O
$R T O_1 O_2$ $R T O_2 O_1$	$\longrightarrow$	R T O O
$T R O_1 O_2$ $T R O_2 O_1$	$\longrightarrow$	T R O O
$O_1 O_2 R T$ $O_2 O_1 T R$	$\longrightarrow$	O O R T
$O_1 R O_2 T$ $O_2 R O_1 T$	$\longrightarrow$	O R O T
$O_1 T O_2 R$ $O_2 T O_1 R$	$\longrightarrow$	O T O R
$O_1 R T O_2$ $O_2 R T O_1$	$\longrightarrow$	O R T O
$O_1 T R O_2$ $O_2 T R O_1$	$\longrightarrow$	O T R O
$O_1 O_2 T R$ $O_2 O_1 T R$	$\longrightarrow$	O O T R

Let us now find the number of ways of rearranging the letters of the word INSTITUTE. In this case there are 9 letters, in which I appears 2 times and T appears 3 times.

Temporarily, let us treat these letters different and name them as I_1, I_2, T_1, T_2, T_3. The number of permutations of 9 different letters, in this case, taken all at a time is 9 !. Consider one such permutation, say, $I_1 NT_1 SI_2 T_2 U E T_3$. Here if I_1, I_2 are not same and T_1, T_2, T_3 are not same, then I_1, I_2 can be arranged in 2! ways and T_1, T_2, T_3 can be arranged in 3! ways. Therefore, 2! × 3! permutations will be just the same permutation corresponding to this chosen permutation $I_1NT_1SI_2T_2UET_3$. Hence, total number of different permutations will be $\frac{9!}{2!\,3!}$

We can state (without proof) the following theorems:

Theorem 3 The number of permutations of n objects, where p objects are of the same kind and rest are all different $= \frac{n!}{p!}$.

In fact, we have a more general theorem.

Theorem 4 The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are of different kind is $\frac{n!}{p_1!\,p_2!\ldots p_k!}$.

Example 9 Find the number of permutations of the letters of the word ALLAHABAD.

Solution Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different.

Therefore, the required number of arrangements $= \frac{9!}{4!\,2!} = \frac{5\times6\times7\times8\times9}{2} = 7560$

Example 10 How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution Here order matters for example 1234 and 1324 are two different numbers. Therefore, there will be as many 4 digit numbers as there are permutations of 9 different digits taken 4 at a time.

Therefore, the required 4 digit numbers $= {}^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024.$

Example 11 How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

Solution Every number between 100 and 1000 is a 3-digit number. We, first, have to count the permutations of 6 digits taken 3 at a time. This number would be $^6P_3$. But, these permutations will include those also where 0 is at the 100's place. For example, 092, 042, . . ., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers has to be subtracted from $^6P_3$ to get the required number. To get the number of such numbers, we fix 0 at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is $^5P_2$. So

The required number $= {}^6P_3 - {}^5P_2 = \frac{6!}{3!} - \frac{5!}{3!}$

$= 4 \times 5 \times 6 - 4 \times 5 = 100$

Example 12 Find the value of n such that

(i) $^nP_5 = 42\ {}^nP_3,\ n > 4$ (ii) $\frac{^nP_4}{^{n-1}P_4} = \frac{5}{3},\ n > 4$

Solution (i) Given that

$^nP_5 = 42\ {}^nP_3$

or $n\ (n-1)\ (n-2)\ (n-3)\ (n-4) = 42\ n(n-1)\ (n-2)$

Since $n > 4$ so $n(n-1)\ (n-2) \neq 0$

Therefore, by dividing both sides by $n(n-1)\ (n-2)$, we get

$(n - 3\ (n-4) = 42$

or $n^2 - 7n - 30 = 0$

or $n^2 - 10n + 3n - 30$

or $(n-10)\ (n+3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so $n = 10$.

(ii) Given that $\frac{^nP_4}{^{n-1}P_4} = \frac{5}{3}$

Therefore $3n\ (n-1)\ (n-2)\ (n-3) = 5(n-1)\ (n-2)\ (n-3)\ (n-4)$

or $3n = 5\ (n-4)$ [as $(n-1)\ (n-2)\ (n-3) \neq 0,\ n > 4$]

or $n = 10$.

Example 13 Find r, if $5\ {}^4P_r = 6\ {}^5P_{r-1}$.

Solution We have $5\ {}^4P_r = 6\ {}^5P_{r-1}$

or $$5\times\frac{4!}{(4-r)!} = 6\times\frac{5!}{(5-r+1)!}$$

or $$\frac{5!}{(4-r)!} = \frac{6\times 5!}{(5-r+1)(5-r)(5-r-1)!}$$

or $(6 - r)(5 - r) = 6$

or $r^2 - 11r + 24 = 0$

or $r^2 - 8r - 3r + 24 = 0$

or $(r - 8)(r - 3) = 0$

or $r = 8$ or $r = 3$.

Hence $r = 8, 3$.

Example 14 Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that

(i) all vowels occur together (ii) all vowels do not occur together.

Solution (i) There are 8 different letters in the word DAUGHTER, in which there are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time. This number would be ${}^6P_6 = 6!$. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time . Hence, by the multiplication principle the required number of permutations = 6 ! × 3 ! = 4320.

(ii) If we have to count those permutations in which all vowels are never together, we first have to find all possible arrangments of 8 letters taken all at a time, which can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together.

Therefore, the required number $8! - 6! \times 3! = 6!\,(7\times8 - 6)$

$= 2 \times 6!\,(28 - 3)$

$= 50 \times 6! = 50 \times 720 = 36000$

Example 15 In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable ?

Solution Total number of discs are 4 + 3 + 2 = 9. Out of 9 discs, 4 are of the first kind (red), 3 are of the second kind (yellow) and 2 are of the third kind (green).

Therefore, the number of arrangements $\frac{9!}{4!\,3!\,2!}=1260$.

Example 16 Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

(i) do the words start with P

(ii) do all the vowels always occur together

(iii) do the vowels never occur together

(iv) do the words begin with I and end in P?

Solution There are 12 letters, of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore

The required number of arrangements $=\frac{12!}{3!\,4!\,2!}=1663200$

(i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters. Therefore, the required number of words starting with P

$$=\frac{11!}{3!\,2!\,4!}=138600.$$

(ii) There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object EEEEI for the time being. This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2 Ds, can be rearranged in $\frac{8!}{3!\,2!}$ ways. Corresponding to each of these arrangements, the 5 vowels E, E, E, E and I can be rearranged in $\frac{5!}{4!}$ ways. Therefore, by multiplication principle, the required number of arrangements

$$=\frac{8!}{3!\,2!}\times\frac{5!}{4!}=16800$$

(iii) The required number of arrangements
= the total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together.

$$= 1663200 - 16800 = 1646400$$

(iv) Let us fix I and P at the extreme ends (I at the left end and P at the right end). We are left with 10 letters.

Hence, the required number of arrangements

$$= \frac{10!}{3!\,2!\,4!} = 12600$$

EXERCISE 6.3

1. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?
2. How many 4-digit numbers are there with no digit repeated?
3. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?
4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?
5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person can not hold more than one position?
6. Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.
7. Find r if (i) ${}^{5}P_r = 2\,{}^{6}P_{r-1}$ (ii) ${}^{5}P_r = {}^{6}P_{r-1}$.
8. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?
9. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.
   (i) 4 letters are used at a time, (ii) all letters are used at a time,
   (iii) all letters are used but first letter is a vowel?
10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?
11. In how many ways can the letters of the word PERMUTATIONS be arranged if the
    (i) words start with P and end with S, (ii) vowels are all together,
    (iii) there are always 4 letters between P and S?

6.4 Combinations

Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X ? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed.

Fig. 6.3

These are XY, YZ and ZX (Fig 6.3).
Here, each selection is called a *combination of 3 different objects taken 2 at a time*. In a combination, the order is not important.

Now consider some more illustrations.

Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of hand shakes. X shaking hands with Y and Y with X will not be two different hand shakes. Here, order is not important. There will be as many hand shakes as there are combinations of 12 different things taken 2 at a time.

Seven points lie on a circle. How many chords can be drawn by joining these points pairwise? There will be as many chords as there are combinations of 7 different things taken 2 at a time.

Now, we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by ${}^nC_r$..

Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., ${}^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations $= {}^4C_2 \times 2!$.

On the other hand, the number of permutations of 4 different things taken 2 at a time $= {}^4P_2$.

Therefore $\quad {}^4P_2 = {}^4C_2 \times 2! \quad$ or $\quad \dfrac{4!}{(4-2)!\,2!} = {}^4C_2$

Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE. Corresponding to each of these ${}^5C_3$ combinations, there are 3! permutations, because, the three objects in each combination can be rearranged in 3 ! ways. Therefore, the total of permutations = $^5C_3 \times 3!$

Therefore $\quad {}^5P_3 = {}^5C_3 \times 3! \quad$ or $\quad \dfrac{5!}{(5-3)!\ 3!} = {}^5C_3$

These examples suggest the following theorem showing relationship between permutaion and combination:

Theorem 5 $^nP_r = {}^nC_r\ r!,\ 0 < r \le n.$

Proof Corresponding to each combination of ^nC_r, we have r ! permutations, because r objects in every combination can be rearranged in r ! ways.

Hence, the total number of permutations of n different things taken r at a time is $^nC_r \times r!$. On the other hand, it is ^nP_r. Thus

$$^nP_r = {}^nC_r \times r!,\ 0 < r \le n.$$

Remarks **1.** From above $\dfrac{n!}{(n-r)!} = {}^nC_r \times r!$, i.e., $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$.

In particular, if $r = n$, $^nC_n = \dfrac{n!}{n!\ 0!} = 1$.

2. We define $^nC_0 = 1$, i.e., the number of combinations of n different things taken nothing at all is considered to be 1. Counting combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there is only one way of doing so. This way we define $^nC_0 = 1$.

3. As $\dfrac{n!}{0!(n-0)!} = 1 = {}^nC_0$, the formula $^nC_r = \dfrac{n!}{r!(n-r)!}$ is applicable for $r = 0$ also.

Hence

$$^nC_r = \frac{n!}{r!\ (n-r)!},\ 0 \le r \le n.$$

4. $\quad {}^nC_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{(n-r)!r!} = {}^nC_r,$

i.e., selecting r objects out of n objects is same as rejecting $(n - r)$ objects.

5. $^nC_a = {^nC_b} \Rightarrow a = b$ or $a = n - b$, i.e., $n = a + b$

Theorem 6 $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

Proof We have $$^nC_r + {^nC_{r-1}} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r \times (r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)\,(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!} \times \frac{n-r+1+r}{r(n-r+1)} = \frac{(n+1)!}{r!(n+1-r)!} = {^{n+1}C_r}$$

Example 17 If $^nC_9 = {^nC_8}$, find $^nC_{17}$.

Solution We have $^nC_9 = {^nC_8}$

i.e., $$\frac{n!}{9!(n-9)!} = \frac{n!}{(n-8)!\,8!}$$

or $$\frac{1}{9} = \frac{1}{n-8} \quad \text{or} \quad n - 8 = 9 \quad \text{or} \quad n = 17$$

Therefore $^nC_{17} = {^{17}C_{17}} = 1$.

Example 18 A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways = $^5C_3 = \frac{5!}{3!\,2!} = \frac{4 \times 5}{2} = 10$.

Now, 1 man can be selected from 2 men in $^2C_1$ ways and 2 women can be selected from 3 women in $^3C_2$ ways. Therefore, the required number of committees

$$= {}^2C_1 \times {}^3C_2 = \frac{2!}{1!\ 1!} \times \frac{3!}{2!\ 1!} = 6.$$

Example 19 What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit,

(ii) four cards belong to four different suits,

(iii) are face cards,

(iv) two are red cards and two are black cards,

(v) cards are of the same colour?

Solution There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore

The required number of ways $= {}^{52}C_4 = \frac{52!}{4!\ 48!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4}$

$$= 270725$$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are ${}^{13}C_4$ ways of choosing 4 clubs, ${}^{13}C_4$ ways of choosing 4 spades and ${}^{13}C_4$ ways of choosing 4 hearts. Therefore

The required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$.

$$= 4 \times \frac{13!}{4!\ 9!} = 2860$$

(ii) There are13 cards in each suit.

Therefore, there are ${}^{13}C_1$ ways of choosing 1 card from 13 cards of diamond, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in ${}^{12}C_4$ ways. Therefore, the required number of ways $= \frac{12!}{4!\ 8!} = 495$.

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

$$= \left(\frac{26!}{2!\ 24!}\right)^2 = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ ways.

Therefore, the required number of ways = ${}^{26}C_4 + {}^{26}C_4$

$$= 2 \times \frac{26!}{4!\ 22!} = 29900.$$

EXERCISE 6.4

1. If ${}^nC_8 = {}^nC_2$, find ${}^nC_2$.
2. Determine n if
   (i) ${}^{2n}C_3 : {}^nC_3 = 12 : 1$ (ii) ${}^{2n}C_3 : {}^nC_3 = 11 : 1$
3. How many chords can be drawn through 21 points on a circle?
4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?
5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.
6. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.
7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?
8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.
9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

Miscellaneous Examples

Example 20 How many words, with or without meaning, each of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE ?

Solution In the word INVOLUTE, there are 4 vowels, namely, I,O,E,Uand 4 consonants, namely, N, V, L and T.

The number of ways of selecting 3 vowels out of 4 = ${}^4C_3 = 4$.

The number of ways of selecting 2 consonants out of 4 = ${}^4C_2 = 6$.

Therefore, the number of combinations of 3 vowels and 2 consonants is $4 \times 6 = 24$.

Now, each of these 24 combinations has 5 letters which can be arranged among themselves in 5 ! ways. Therefore, the required number of different words is $24 \times 5\,! = 2880$.

Example 21 A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girl ? (ii) at least one boy and one girl ? (iii) at least 3 girls ?

Solution (i) Since, the team will not include any girl, therefore, only boys are to be selected. 5 boys out of 7 boys can be selected in ${}^7C_5$ ways. Therefore, the required

number of ways $= {}^7C_5 = \dfrac{7!}{5!\ 2!} = \dfrac{6 \times 7}{2} = 21$

(ii) Since, at least one boy and one girl are to be there in every team. Therefore, the team can consist of

(a) 1 boy and 4 girls (b) 2 boys and 3 girls

(c) 3 boys and 2 girls (d) 4 boys and 1 girl.

1 boy and 4 girls can be selected in ${}^7C_1 \times {}^4C_4$ ways.

2 boys and 3 girls can be selected in ${}^7C_2 \times {}^4C_3$ ways.

3 boys and 2 girls can be selected in ${}^7C_3 \times {}^4C_2$ ways.

4 boys and 1 girl can be selected in ${}^7C_4 \times {}^4C_1$ ways.

Therefore, the required number of ways

$= {}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1$

$= 7 + 84 + 210 + 140 = 441$

(iii) Since, the team has to consist of at least 3 girls, the team can consist of

(a) 3 girls and 2 boys, or (b) 4 girls and 1 boy.

Note that the team cannot have all 5 girls, because, the group has only 4 girls.

3 girls and 2 boys can be selected in ${}^4C_3 \times {}^7C_2$ ways.

4 girls and 1 boy can be selected in ${}^4C_4 \times {}^7C_1$ ways.

Therefore, the required number of ways

$= {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 84 + 7 = 91$

Example 22 Find the number of words with or without meaning which can be made using all the letters of the word AGAIN. If these words are written as in a dictionary, what will be the 50^{th} word?

Solution There are 5 letters in the word AGAIN, in which A appears 2 times. Therefore, the required number of words $= \frac{5!}{2!} = 60$.

To get the number of words starting with A, we fix the letter A at the extreme left position, we then rearrange the remaining 4 letters taken all at a time. There will be as many arrangements of these 4 letters taken 4 at a time as there are permutations of 4 different things taken 4 at a time. Hence, the number of words starting with A $= 4! = 24$. Then, starting with G, the number of words $= \frac{4!}{2!} = 12$ as after placing G at the extreme left position, we are left with the letters A, A, I and N. Similarly, there are 12 words starting with the next letter I. Total number of words so far obtained $= 24 + 12 + 12 = 48$.

The 49^{th} word is NAAGI. The 50^{th} word is NAAIG.

Example 23 How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4?

Solution Since, 1000000 is a 7-digit number and the number of digits to be used is also 7. Therefore, the numbers to be counted will be 7-digit only. Also, the numbers have to be greater than 1000000, so they can begin either with 1, 2 or 4.

The number of numbers beginning with $1 = \frac{6!}{3!\,2!} = \frac{4 \times 5 \times 6}{2} = 60$, as when 1 is fixed at the extreme left position, the remaining digits to be rearranged will be 0, 2, 2, 2, 4, 4, in which there are 3, 2*s* and 2, 4*s*.

Total numbers begining with 2

$$= \frac{6!}{2!\,2!} = \frac{3 \times 4 \times 5 \times 6}{2} = 180$$

and total numbers begining with $4 = \frac{6!}{3!} = 4 \times 5 \times 6 = 120$

Therefore, the required number of numbers $= 60 + 180 + 120 = 360$.

Alternative Method

The number of 7-digit arrangements, clearly, $\frac{7!}{3!\ 2!} = 420$. But, this will include those numbers also, which have 0 at the extreme left position. The number of such arrangements $\frac{6!}{3!\ 2!}$ (by fixing 0 at the extreme left position) = 60.

Therefore, the required number of numbers = 420 – 60 = 360.

> **Note** If one or more than one digits given in the list is repeated, it will be understood that in any number, the digits can be used as many times as is given in the list, e.g., in the above example 1 and 0 can be used only once whereas 2 and 4 can be used 3 times and 2 times, respectively.

Example 24 In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?

Solution Let us first seat the 5 girls. This can be done in 5! ways. For each such arrangement, the three boys can be seated only at the cross marked places.

$$\times G \times G \times G \times G \times G \times.$$

There are 6 cross marked places and the three boys can be seated in ${}^6P_3$ ways. Hence, by multiplication principle, the total number of ways

$$= 5! \times {}^6P_3 = 5! \times \frac{6!}{3!}$$
$$= 4 \times 5 \times 2 \times 3 \times 4 \times 5 \times 6 = 14400.$$

Miscellaneous Exercise on Chapter 6

1. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER ?
2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonants occur together?
3. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:

   (i) exactly 3 girls ? (ii) atleast 3 girls ? (iii) atmost 3 girls ?
4. If the different permutations of all the letter of the word EXAMINATION are

listed as in a dictionary, how many words are there in this list before the first word starting with E ?

5. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9 which are divisible by 10 and no digit is repeated ?
6. The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?
7. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions ?
8. Determine the number of 5-card combinations out of a deck of 52 cards if each selection of 5 cards has exactly one king.
9. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ?
10. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen ?
11. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together ?

Summary

- *Fundamental principle of counting* If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.
- The number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by ${}^nP_r$ and is given by ${}^nP_r = \dfrac{n!}{(n-r)!}$, where $0 \le r \le n$.
- $n! = 1 \times 2 \times 3 \times ...\times n$
- $n! = n \times (n-1)!$
- The number of permutations of n different things, taken r at a time, where repeatition is allowed, is n^r.
- The number of permutations of n objects taken all at a time, where p_1 objects

are of first kind, p_2 objects are of the second kind, ..., p_k objects are of the k^{th} kind and rest, if any, are all different is $\frac{n!}{p_1!\, p_2! \ldots p_k!}$.

- The number of combinations of n different things taken r at a time, denoted by ^nC_r, is given by $^nC_r = = \frac{n!}{r!(n-r)!}$, $0 \le r \le n$.

Historical Note

The concepts of permutations and combinations can be traced back to the advent of Jainism in India and perhaps even earlier. The credit, however, goes to the Jains who treated its subject matter as a self-contained topic in mathematics, under the name *Vikalpa*.

Among the Jains, *Mahavira*, (around 850) is perhaps the world's first mathematician credited with providing the general formulae for permutations and combinations.

In the 6th century B.C., *Sushruta*, in his medicinal work, *Sushruta Samhita*, asserts that 63 combinations can be made out of 6 different tastes, taken one at a time, two at a time, etc. *Pingala*, a Sanskrit scholar around third century B.C., gives the method of determining the number of combinations of a given number of letters, taken one at a time, two at a time, etc. in his work *Chhanda Sutra*. *Bhaskaracharya* (born 1114) treated the subject matter of permutations and combinations under the name *Anka Pasha* in his famous work *Lilavati*. In addition to the general formulae for ^nC_r and ^nP_r already provided by *Mahavira*, *Bhaskaracharya* gives several important theorems and results concerning the subject.

Outside India, the subject matter of permutations and combinations had its humble beginnings in China in the famous book I–King (Book of changes). It is difficult to give the approximate time of this work, since in 213 B.C., the emperor had ordered all books and manuscripts in the country to be burnt which fortunately was not completely carried out. Greeks and later Latin writers also did some scattered work on the theory of permutations and combinations.

Some Arabic and Hebrew writers used the concepts of permutations and combinations in studying astronomy. *Rabbi ben Ezra*, for instance, determined the number of combinations of known planets taken two at a time, three at a time and so on. This was around 1140. It appears that *Rabbi ben Ezra* did not know the formula for ${}^nC_r$. However, he was aware that ${}^nC_r = {}^nC_{n-r}$ for specific values n and r. In 1321, *Levi Ben Gerson*, another Hebrew writer came up with the formulae for ${}^nP_r$, ${}^nP_n$ *and the general formula for* ${}^nC_r$.

The first book which gives a complete treatment of the subject matter of permutations and combinations is Ars Conjectandi written by a Swiss, *Jacob Bernoulli* (1654 – 1705), posthumously published in 1713. This book contains essentially the theory of permutations and combinations as is known today.

Chapter 7

BINOMIAL THEOREM

❖ *Mathematics is a most exact science and its conclusions are capable of absolute proofs. – C.P. STEINMETZ* ❖

7.1 Introduction

In earlier classes, we have learnt how to find the squares and cubes of binomials like $a + b$ and $a - b$. Using them, we could evaluate the numerical values of numbers like $(98)^2 = (100 - 2)^2$, $(999)^3 = (1000 - 1)^3$, etc. However, for higher powers like $(98)^5$, $(101)^6$, etc., the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand $(a + b)^n$, where n is an integer or a rational number. In this Chapter, we study binomial theorem for positive integral indices only.

Blaise Pascal (1623-1662)

7.2 Binomial Theorem for Positive Integral Indices

Let us have a look at the following identities done earlier:

$$(a+b)^0 = 1 \qquad a + b \neq 0$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = (a+b)^3\,(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

In these expansions, we observe that

(i) The total number of terms in the expansion is one more than the index. For example, in the expansion of $(a + b)^2$, number of terms is 3 whereas the index of $(a + b)^2$ is 2.

(ii) Powers of the first quantity 'a' go on decreasing by 1 whereas the powers of the second quantity 'b' increase by 1, in the successive terms.

(iii) In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a + b$.

We have shown that for all real x, $\sin^2 x + \cos^2 x = 1$

It follows that

$$1 + \tan^2 x = \sec^2 x \qquad \text{(why?)}$$

$$1 + \cot^2 x = \text{cosec}^2 x \qquad \text{(why?)}$$

In earlier classes, we have discussed the values of trigonometric ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometric ratios studied in earlier classes. Thus, we have the following table:

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively.

3.3.1 *Sign of trigonometric functions*

Let P (a, b) be a point on the unit circle with centre at the origin such that $\angle AOP = x$. If $\angle AOQ = -x$, then the coordinates of the point Q will be $(a, -b)$ (Fig 3.7). Therefore

$\cos(-x) = \cos x$

and $\sin(-x) = -\sin x$

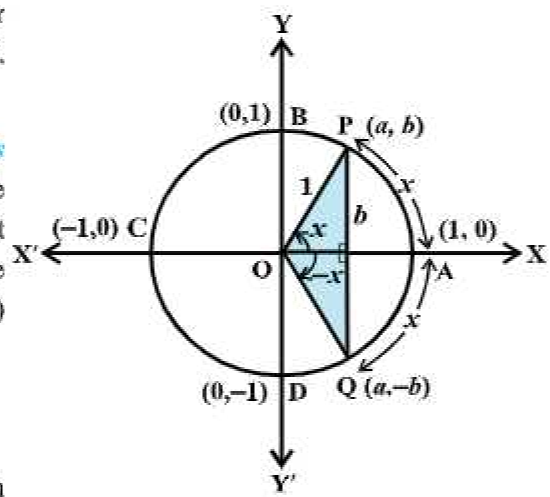


Fig 3.7

Since for every point P (a, b) on the unit circle, $-1 \le a \le 1$ and

$-1 \leq b \leq 1$, we have $-1 \leq \cos x \leq 1$ and $-1 \leq \sin x \leq 1$ for all x. We have learnt in previous classes that in the first quadrant ($0 < x < \frac{\pi}{2}$) a and b are both positive, in the second quadrant ($\frac{\pi}{2} < x < \pi$) a is negative and b is positive, in the third quadrant ($\pi < x < \frac{3\pi}{2}$) a and b are both negative and in the fourth quadrant ($\frac{3\pi}{2} < x < 2\pi$) a is positive and b is negative. Therefore, $\sin x$ is positive for $0 < x < \pi$, and negative for $\pi < x < 2\pi$. Similarly, $\cos x$ is positive for $0 < x < \frac{\pi}{2}$, negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$. Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\text{cosec } x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.3.2 ***Domain and range of trigonometric functions*** From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x,

$$-1 \leq \sin x \leq 1 \text{ and } -1 \leq \cos x \leq 1$$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval $[-1, 1]$, i.e., $-1 \leq y \leq 1$.

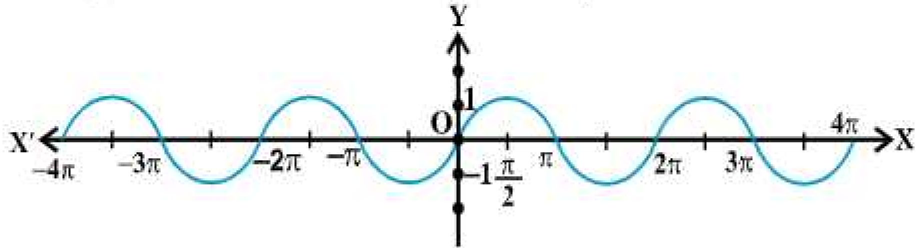
Since cosec $x = \frac{1}{\sin x}$, the domain of y = cosec x is the set $\{ x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \geq 1$ or $y \leq -1\}$. Similarly, the domain of $y = \sec x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \leq -1$ or $y \geq 1\}$. The domain of $y = \tan x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set of all real numbers. The domain of $y = \cot x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and the range is the set of all real numbers.

We further observe that in the first quadrant, as x increases from 0 to $\frac{\pi}{2}$, $\sin x$ increases from 0 to 1, as x increases from $\frac{\pi}{2}$ to π, $\sin x$ decreases from 1 to 0. In the third quadrant, as x increases from π to $\frac{3\pi}{2}$, $\sin x$ decreases from 0 to –1and finally, in the fourth quadrant, $\sin x$ increases from –1 to 0 as x increases from $\frac{3\pi}{2}$ to 2π. Similarly, we can discuss the behaviour of other trigonometric functions. In fact, we have the following table:

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0
cos	decreases from 1 to 0	decreases from 0 to – 1	increases from –1 to 0	increases from 0 to 1
tan	increases from 0 to ∞	increases from –∞to 0	increases from 0 to ∞	increases from –∞to 0
cot	decreases from ∞ to 0	decreases from 0 to–∞	decreases from ∞ to 0	decreases from 0to –∞
sec	increases from 1 to ∞	increases from –∞to–1	decreases from –1to–∞	decreases from ∞ to 1
cosec	decreases from ∞ to 1	increases from 1 to ∞	increases from –∞to–1	decreases from–1to–∞

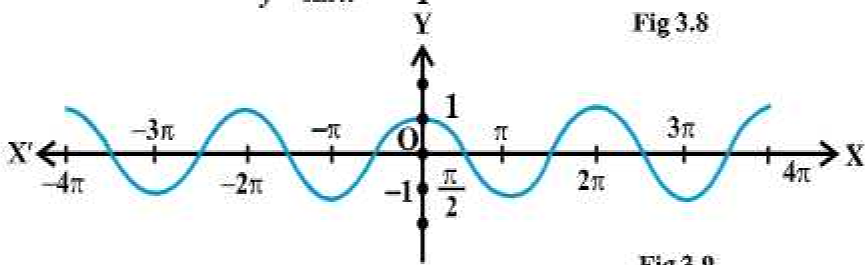
Remark In the above table, the statement $\tan x$ increases from 0 to ∞ (infinity) for $0 < x < \frac{\pi}{2}$ simply means that $\tan x$ increases as x increases for $0 < x < \frac{\pi}{2}$ and assumes arbitraily large positive values as x approaches to $\frac{\pi}{2}$. Similarly, to say that cosec x decreases from –1 to – ∞ (minus infinity) in the fourth quadrant means that cosec x decreases for $x \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as x approaches to 2π. The symbols ∞ and – ∞ simply specify certain types of behaviour of functions and variables.

We have already seen that values of sin x and cos x repeats after an interval of 2π. Hence, values of cosec x and sec x will also repeat after an interval of 2π. We



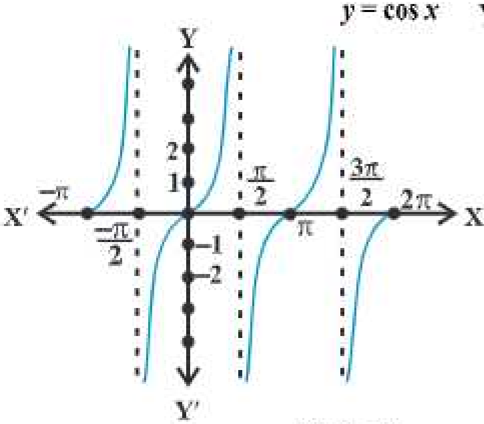
y = sin *x*

Fig 3.8

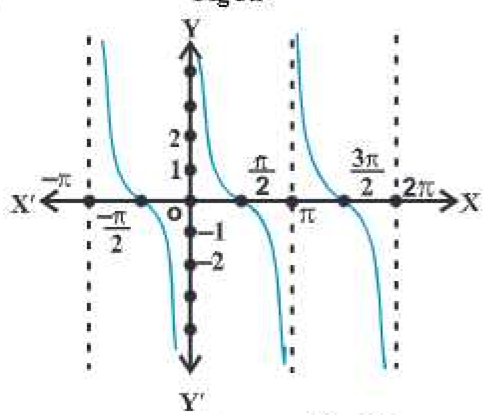


y = cos *x*

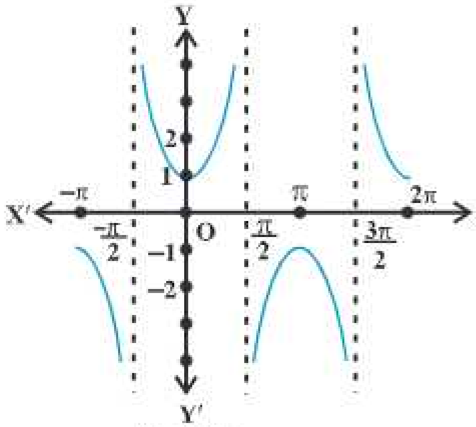
Fig 3.9



y = tan *x* Fig 3.10

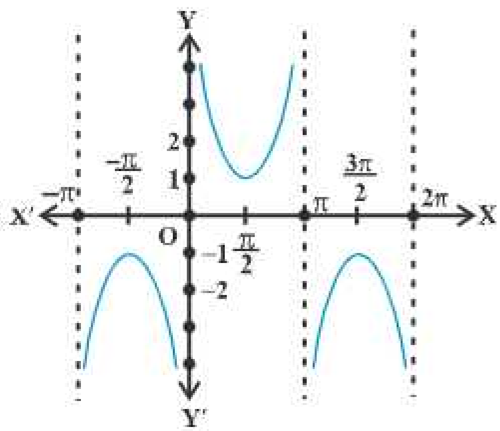


y = cot *x* Fig 3.11



$y = \sec x$

Fig 3.12



$y = \text{cosec}\, x$

Fig 3.13

shall see in the next section that $\tan(\pi + x) = \tan x$. Hence, values of $\tan x$ will repeat after an interval of π. Since $\cot x$ is reciprocal of $\tan x$, its values will also repeat after an interval of π. Using this knowledge and behaviour of trigonometic functions, we can sketch the graph of these functions. The graph of these functions are given above:

Example 6 If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now $\sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\sin x = \pm \frac{4}{5}$

Since x lies in third quadrant, $\sin x$ is negative. Therefore

$$\sin x = -\frac{4}{5}$$

which also gives

$$\text{cosec}\, x = -\frac{5}{4}$$

Further, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \text{ and } \cot x = \frac{\cos x}{\sin x} = \frac{3}{4}.$$

Example 7 If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution Since $\cot x = -\frac{5}{12}$, we have $\tan x = -\frac{12}{5}$

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$

Hence $\sec x = \pm \frac{13}{5}$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$$\sec x = -\frac{13}{5},$$

which also gives

$$\cos x = -\frac{5}{13}$$

Further, we have

$$\sin x = \tan x \cos x = \left(-\frac{12}{5}\right) \times \left(-\frac{5}{13}\right) = \frac{12}{13}$$

and $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}.$

Example 8 Find the value of $\sin \frac{31\pi}{3}$.

Solution We know that values of $\sin x$ repeats after an interval of 2π. Therefore

$$\sin \frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 9 Find the value of cos (–1710°).

Solution We know that values of cos x repeats after an interval of 2π or 360°. Therefore, $\cos(-1710°) = \cos(-1710° + 5 \times 360°)$

$$= \cos(-1710° + 1800°) = \cos 90° = 0.$$

EXERCISE 3.2

Find the values of other five trigonometric functions in Exercises 1 to 5.

1. $\cos x = -\frac{1}{2}$, x lies in third quadrant.
2. $\sin x = \frac{3}{5}$, x lies in second quadrant.
3. $\cot x = \frac{3}{4}$, x lies in third quadrant.
4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.
5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Find the values of the trigonometric functions in Exercises 6 to 10.

6. sin 765°
7. cosec (– 1410°)
8. $\tan \frac{19\pi}{3}$
9. $\sin (-\frac{11\pi}{3})$
10. $\cot (-\frac{15\pi}{4})$

3.4 Trigonometric Functions of Sum and Difference of Two Angles

In this Section, we shall derive expressions for trigonometric functions of the sum and difference of two numbers (angles) and related expressions. The basic results in this connection are called *trigonometric identities*. We have seen that

1. **$\sin(-x) = -\sin x$**
2. **$\cos(-x) = \cos x$**

We shall now prove some more results:

3. $\cos(x+y) = \cos x \cos y - \sin x \sin y$

Consider the unit circle with centre at the origin. Let x be the angle P_4OP_1 and y be the angle P_1OP_2. Then $(x + y)$ is the angle P_4OP_2. Also let $(-y)$ be the angle P_4OP_3. Therefore, P_1, P_2, P_3 and P_4 will have the coordinates $P_1(\cos x, \sin x)$, $P_2[\cos(x+y), \sin(x+y)]$, $P_3[\cos(-y), \sin(-y)]$ and $P_4(1, 0)$ (Fig 3.14).

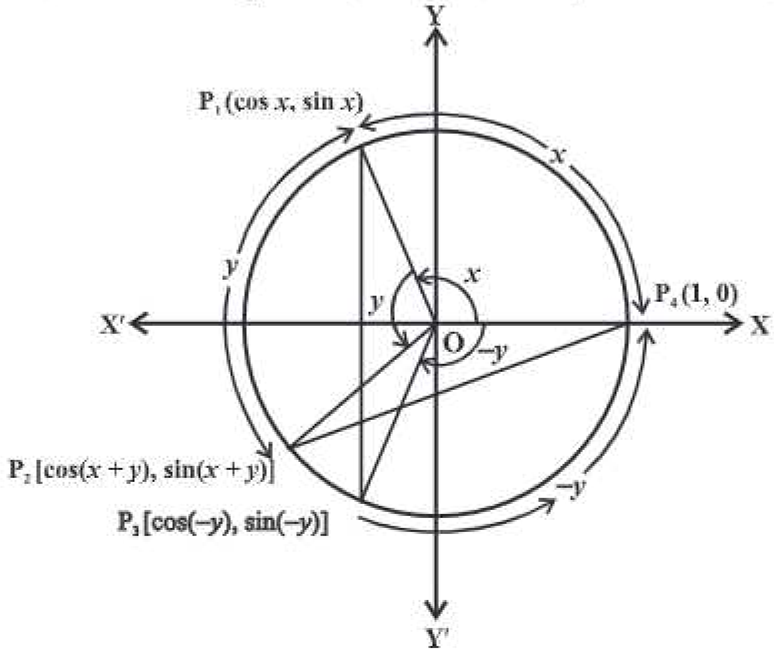


Fig 3.14

Consider the triangles P_1OP_3 and P_2OP_4. They are congruent (Why?). Therefore, P_1P_3 and P_2P_4 are equal. By using distance formula, we get

$$
\begin{aligned}
P_1P_3^2 &= [\cos x - \cos(-y)]^2 + [\sin x - \sin(-y)]^2 \\
&= (\cos x - \cos y)^2 + (\sin x + \sin y)^2 \\
&= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y \\
&= 2 - 2(\cos x \cos y - \sin x \sin y) \qquad \text{(Why?)}
\end{aligned}
$$

Also,

$$
\begin{aligned}
P_2P_4^2 &= [1 - \cos(x+y)]^2 + [0 - \sin(x+y)]^2 \\
&= 1 - 2\cos(x+y) + \cos^2(x+y) + \sin^2(x+y) \\
&= 2 - 2\cos(x+y)
\end{aligned}
$$

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$.

Therefore, $2 - 2 (\cos x \cos y - \sin x \sin y) = 2 - 2 \cos (x + y)$.

Hence $\cos (x + y) = \cos x \cos y - \sin x \sin y$

4. $\cos (x - y) = \cos x \cos y + \sin x \sin y$

Replacing y by $-y$ in identity 3, we get

$$\cos (x + (-y)) = \cos x \cos (-y) - \sin x \sin (-y)$$

or $\cos (x - y) = \cos x \cos y + \sin x \sin y$

5. $\cos (\frac{\pi}{2} - x) = \sin x$

If we replace x by $\frac{\pi}{2}$ and y by x in Identity (4), we get

$$\cos (\frac{\pi}{2} - x) = \cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x = \sin x.$$

6. $\sin (\frac{\pi}{2} - x) = \cos x$

Using the Identity 5, we have

$$\sin (\frac{\pi}{2} - x) = \cos \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \cos x.$$

7. $\sin (x + y) = \sin x \cos y + \cos x \sin y$

We know that

$$\sin (x + y) = \cos \left(\frac{\pi}{2} - (x + y)\right) = \cos \left((\frac{\pi}{2} - x) - y\right)$$

$$= \cos (\frac{\pi}{2} - x) \cos y + \sin (\frac{\pi}{2} - x) \sin y$$

$$= \sin x \cos y + \cos x \sin y$$

8. $\sin (x - y) = \sin x \cos y - \cos x \sin y$

If we replace y by $-y$, in the Identity 7, we get the result.

9. By taking suitable values of x and y in the identities 3, 4, 7 and 8, we get the following results:

$$\cos (\frac{\pi}{2} + x) = -\sin x \qquad \sin (\frac{\pi}{2} + x) = \cos x$$

$$\cos (\pi - x) = -\cos x \qquad \sin (\pi - x) = \sin x$$

$$\cos(\pi + x) = -\cos x \qquad \sin(\pi + x) = -\sin x$$
$$\cos(2\pi - x) = \cos x \qquad \sin(2\pi - x) = -\sin x$$

Similar results for $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ can be obtained from the results of $\sin x$ and $\cos x$.

10. If none of the angles x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Since none of the x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, it follows that $\cos x$, $\cos y$ and $\cos(x + y)$ are non-zero. Now

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Dividing numerator and denominator by $\cos x \cos y$, we have

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

11. $$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we replace y by $-y$ in Identity 10, we get

$$\tan(x - y) = \tan[x + (-y)]$$

$$= \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

12. If none of the angles x, y and $(x + y)$ is a multiple of π, then

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

Since, none of the x, y and $(x + y)$ is multiple of π, we find that $\sin x \; \sin y$ and $\sin (x + y)$ are non-zero. Now,

$$\cot (x + y) = \frac{\cos (x + y)}{\sin (x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

Dividing numerator and denominator by $\sin x \sin y$, we have

$$\cot (x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

13. $\mathbf{\cot (x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}}$ if none of angles x, y and $x-y$ is a multiple of π

If we replace y by $-y$ in identity 12, we get the result

14. $\mathbf{\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}}$

We know that

$$\cos (x + y) = \cos x \; \cos y - \sin x \sin y$$

Replacing y by x, we get

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$

Again, $\cos 2x = \cos^2 x - \sin^2 x$
$$= 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x.$$

We have $\cos 2x = \cos^2 x - \sin^2 x = \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$

Dividing numerator and denominator by $\cos^2 x$, we get

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}, \quad x \neq n\pi + \frac{\pi}{2}, \text{ where } n \text{ is an integer}$$

15. $\mathbf{\sin 2x = 2 \sin x \cos x = \frac{2\tan x}{1 + \tan^2 x}}$ $x \neq n\pi + \frac{\pi}{2}$, where n is an integer

We have

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

Replacing y by x, we get $\sin 2x = 2 \sin x \cos x$.

Again $\sin 2x = \frac{2\sin x \cos x}{\cos^2 x + \sin^2 x}$

Dividing each term by $\cos^2 x$, we get

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

16. **$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$** if $2x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We know that

$$\tan (x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Replacing y by x , we get $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

17. **$\sin 3x = 3 \sin x - 4 \sin^3 x$**

We have,

$$\begin{aligned}\sin 3x &= \sin (2x + x)\\ &= \sin 2x \cos x + \cos 2x \sin x\\ &= 2 \sin x \cos x \cos x + (1 - 2\sin^2 x) \sin x\\ &= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x\\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x\\ &= 3 \sin x - 4 \sin^3 x\end{aligned}$$

18. **$\cos 3x = 4 \cos^3 x - 3 \cos x$**

We have,

$$\begin{aligned}\cos 3x &= \cos (2x + x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= (2\cos^2 x - 1) \cos x - 2\sin x \cos x \sin x\\ &= (2\cos^2 x - 1) \cos x - 2\cos x (1 - \cos^2 x)\\ &= 2\cos^3 x - \cos x - 2\cos x + 2 \cos^3 x\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

19. **$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$** if $3x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We have $\tan 3x = \tan (2x + x)$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \dfrac{2\tan x \,.\, \tan x}{1-\tan^2 x}}$$

$$= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

20. **(i)** $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

(ii) $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

(iii) $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

(iv) $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

We know that

$\cos(x+y) = \cos x\cos y - \sin x\sin y$ ... (1)

and $\cos(x-y) = \cos x\cos y + \sin x\sin y$ ... (2)

Adding and subtracting (1) and (2), we get

$\cos(x+y) + \cos(x-y) = 2\cos x\cos y$ ... (3)

and $\cos(x+y) - \cos(x-y) = -2\sin x\sin y$ ... (4)

Further $\sin(x+y) = \sin x\cos y + \cos x\sin y$ ... (5)

and $\sin(x-y) = \sin x\cos y - \cos x\sin y$ ... (6)

Adding and subtracting (5) and (6), we get

$\sin(x+y) + \sin(x-y) = 2\sin x\cos y$ ... (7)

$\sin(x+y) - \sin(x-y) = 2\cos x\sin y$ ... (8)

Let $x + y = \theta$ and $x - y = \phi$. Therefore

$$x = \left(\frac{\theta+\phi}{2}\right) \text{and } y = \left(\frac{\theta-\phi}{2}\right)$$

Substituting the values of x and y in (3), (4), (7) and (8), we get

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Since θ and ϕ can take any real values, we can replace θ by x and ϕ by y.
Thus, we get

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\ ;\ \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2},$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\ ;\ \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$$

Remark As a part of identities given in 20, we can prove the following results:

21. **(i)** $\mathbf{2\cos x\cos y = \cos(x+y) + \cos(x-y)}$
(ii) $\mathbf{-2\sin x\sin y = \cos(x+y) - \cos(x-y)}$
(iii) $\mathbf{2\sin x\cos y = \sin(x+y) + \sin(x-y)}$
(iv) $\mathbf{2\cos x\sin y = \sin(x+y) - \sin(x-y)}$.

Example 10 Prove that

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$$

Solution We have

$$\text{L.H.S.} = 3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$$

$$= 3\times\frac{1}{2}\times 2 - 4\sin\left(\pi - \frac{\pi}{6}\right)\times 1 = 3 - 4\sin\frac{\pi}{6}$$

$$= 3 - 4\times\frac{1}{2} = 1 = \text{R.H.S.}$$

Example 11 Find the value of sin 15°.

Solution We have

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$

$$= \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 12 Find the value of $\tan\frac{13\pi}{12}$.

Solution We have

$$\tan \frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right) = \tan \frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$$

Example 13 Prove that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Solution We have

$$\text{L.H.S.} \quad = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

Dividing the numerator and denominator by $\cos x \cos y$, we get

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Example 14 Show that

$$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

Solution We know that $3x = 2x + x$

Therefore, $\tan 3x = \tan(2x + x)$

or $\quad \tan 3x = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

or $\quad \tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$

or $\quad \tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$

or $\quad \tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x.$

Example 15 Prove that

$$\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x$$

Solution Using the Identity 20(i), we have

$$\text{L.H.S.} = \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$$

$$= 2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-(\frac{\pi}{4}-x)}{2}\right)$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R.H.S.}$$

Example 16 Prove that $\dfrac{\cos 7x+\cos 5x}{\sin 7x-\sin 5x}=\cot x$

Solution Using the Identities 20 (i) and 20 (iv), we get

$$\text{L.H.S.} = \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x = \text{R.H.S.}$$

Example 17 Prove that $=\dfrac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x}=\tan x$

Solution We have

$$\text{L.H.S.} = \frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \frac{\sin 5x+\sin x-2\sin 3x}{\cos 5x-\cos x}$$

$$= \frac{2\sin 3x\cos 2x-2\sin 3x}{-2\sin 3x\sin 2x} = -\frac{\sin 3x\,(\cos 2x-1)}{\sin 3x\sin 2x}$$

$$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x = \text{R.H.S.}$$

EXERCISE 3.3

Prove that:

1. $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$

2. $2\sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$

3. $\cot^2 \frac{\pi}{6} + \operatorname{cosec} \frac{5\pi}{6} + 3\tan^2 \frac{\pi}{6} = 6$

4. $2\sin^2 \frac{3\pi}{4} + 2\cos^2 \frac{\pi}{4} + 2\sec^2 \frac{\pi}{3} = 10$

5. Find the value of:

   (i) $\sin 75^\circ$ (ii) $\tan 15^\circ$

Prove the following:

6. $\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin(x + y)$

7. $\dfrac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\dfrac{1 + \tan x}{1 - \tan x}\right)^2$

8. $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$

9. $\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right] = 1$

10. $\sin(n + 1)x \sin(n + 2)x + \cos(n + 1)x \cos(n + 2)x = \cos x$

11. $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$

12. $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

13. $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

14. $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

15. $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$

16. $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$

17. $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

18. $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan \dfrac{x - y}{2}$

19. $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

20. $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$

21. $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

22. $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

23. $\tan 4x = \dfrac{4\tan x\,(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$ **24.** $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

25. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

Miscellaneous Examples

Example 18 If $\sin x = \dfrac{3}{5}$, $\cos y = -\dfrac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin (x + y)$.

Solution We know that

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \qquad \dots (1)$$

Now $\cos^2 x = 1 - \sin^2 x = 1 - \dfrac{9}{25} = \dfrac{16}{25}$

Therefore $\cos x = \pm\dfrac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\cos x = -\dfrac{4}{5}$

Now $\sin^2 y = 1 - \cos^2 y = 1 - \dfrac{144}{169} = \dfrac{25}{169}$

i.e. $\sin y = \pm\dfrac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \dfrac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get

$$\sin(x + y) = \frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13} = -\frac{36}{65}-\frac{20}{65}=-\frac{56}{65}.$$

Example 19 Prove that

$$\cos 2x \cos\frac{x}{2} - \cos 3x \cos\frac{9x}{2} = \sin 5x \sin\frac{5x}{2}.$$

Solution We have

L.H.S. $= \frac{1}{2}\left[2\cos 2x \cos\frac{x}{2} - 2\cos\frac{9x}{2}\cos 3x\right]$

$$= \frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right)+\cos\left(2x-\frac{x}{2}\right)-\cos\left(\frac{9x}{2}+3x\right)-\cos\left(\frac{9x}{2}-3x\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5x}{2}+\cos\frac{3x}{2}-\cos\frac{15x}{2}-\cos\frac{3x}{2}\right] = \frac{1}{2}\left[\cos\frac{5x}{2}-\cos\frac{15x}{2}\right]$$

$$= \frac{1}{2}\left[-2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

$$= -\sin 5x \sin\left(-\frac{5x}{2}\right) = \sin 5x \sin\frac{5x}{2} = \text{R.H.S.}$$

Example 20 Find the value of $\tan\frac{\pi}{8}$.

Solution Let $x = \frac{\pi}{8}$. Then $2x = \frac{\pi}{4}$.

Now $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

or $\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

Let $y = \tan\frac{\pi}{8}$. Then $1 = \frac{2y}{1-y^2}$

or $y^2 + 2y - 1 = 0$

Therefore $y = \frac{-2\pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $\frac{\pi}{8}$ lies in the first quadrant, $y = \tan\frac{\pi}{8}$ is positve. Hence

$$\tan\frac{\pi}{8} = \sqrt{2} - 1.$$

Example 21 If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Solution Since $\pi < x < \frac{3\pi}{2}$, $\cos x$ is negative.

Also $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$.

Therefore, $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{9}{16} = \frac{25}{16}$

Therefore $\cos^2 x = \frac{16}{25}$ or $\cos x = -\frac{4}{5}$ (Why?)

Now $2\sin^2\frac{x}{2} = 1 - \cos x = 1 + \frac{4}{5} = \frac{9}{5}$.

Therefore $\sin^2\frac{x}{2} = \frac{9}{10}$

or $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ (Why?)

Again $2\cos^2\frac{x}{2} = 1 + \cos x = 1 - \frac{4}{5} = \frac{1}{5}$

Therefore $\cos^2\frac{x}{2} = \frac{1}{10}$

or $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$ (Why?)

Hence $$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3}{\sqrt{10}}\times\left(\frac{-\sqrt{10}}{1}\right) = -3.$$

Example 22 Prove that $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right)+\cos^2\left(x-\frac{\pi}{3}\right)=\frac{3}{2}$

Solution We have

$$\text{L.H.S.} = \frac{1+\cos 2x}{2}+\frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2}+\frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}.$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-\cos 2x\right]=\frac{3}{2} = \text{R.H.S.}$$

Miscellaneous Exercise on Chapter 3

Prove that:

1. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

2. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

3. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\frac{x+y}{2}$

4. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \frac{x-y}{2}$

5. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$

6. $\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

7. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos \frac{x}{2} \cos \frac{3x}{2}$

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ in each of the following :

8. $\tan x = -\frac{4}{3}$, x in quadrant II

9. $\cos x = -\frac{1}{3}$, x in quadrant III

10. $\sin x = \frac{1}{4}$, x in quadrant II

Summary

- If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\,\theta$
- Radian measure $= \frac{\pi}{180} \times$ Degree measure
- Degree measure $= \frac{180}{\pi} \times$ Radian measure
- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \text{cosec}^2 x$
- $\cos (2n\pi + x) = \cos x$
- $\sin (2n\pi + x) = \sin x$
- $\sin (-x) = -\sin x$
- $\cos (-x) = \cos x$
- $\cos (x + y) = \cos x \cos y - \sin x \sin y$
- $\cos (x - y) = \cos x \cos y + \sin x \sin y$
- $\cos (\frac{\pi}{2} - x) = \sin x$

◆ $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

◆ $\sin(x + y) = \sin x \cos y + \cos x \sin y$

◆ $\sin(x - y) = \sin x \cos y - \cos x \sin y$

◆ $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$ $\qquad$ $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

$\cos(\pi - x) = -\cos x$ $\qquad$ $\sin(\pi - x) = \sin x$

$\cos(\pi + x) = -\cos x$ $\qquad$ $\sin(\pi + x) = -\sin x$

$\cos(2\pi - x) = \cos x$ $\qquad$ $\sin(2\pi - x) = -\sin x$

◆ If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

◆ $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

◆ If none of the angles x, y and $(x \pm y)$ is a multiple of π, then

$$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

◆ $\cot(x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

◆ $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

◆ $\sin 2x = 2 \sin x \cos x = \dfrac{2 \tan x}{1 + \tan^2 x}$

◆ $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

◆ $\sin 3x = 3\sin x - 4\sin^3 x$

◆ $\cos 3x = 4\cos^3 x - 3\cos x$

- $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

- (i) $\cos x + \cos y = 2\cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

  (ii) $\cos x - \cos y = -2\sin \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

  (iii) $\sin x + \sin y = 2\sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

  (iv) $\sin x - \sin y = 2\cos \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

- (i) $2\cos x \cos y = \cos (x+y) + \cos (x-y)$

  (ii) $-2\sin x \sin y = \cos (x+y) - \cos (x-y)$

  (iii) $2\sin x \cos y = \sin (x+y) + \sin (x-y)$

  (iv) $2\cos x \sin y = \sin (x+y) - \sin (x-y)$.

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhatta (476), Brahmagupta (598), Bhaskara I (600) and Bhaskara II (1114) got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents the main contribution of the *siddhantas* (Sanskrit astronomical works) to the history of mathematics.

Bhaskara I (about 600) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* (period) contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., are given by Bhaskara II.

The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The names of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{H}{S} = \frac{h}{s} = \tan \text{(sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

Chapter **4**

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

❖ *Mathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics. – GAUSS* ❖

4.1 Introduction

In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution as $x^2 + 1 = 0$ gives $x^2 = -1$ and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation $x^2 = -1$. In fact, the main objective is to solve the equation $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers.

W. R. Hamilton (1805-1865)

4.2 Complex Numbers

Let us denote $\sqrt{-1}$ by the symbol i. Then, we have $i^2 = -1$. This means that i is a solution of the equation $x^2 + 1 = 0$.

A number of the form $a + ib$, where a and b are real numbers, is defined to be a complex number. For example, $2 + i3$, $(-1) + i\sqrt{3}$, $4 + i\left(\frac{-1}{11}\right)$ are complex numbers.

For the complex number $z = a + ib$, a is called the *real part*, denoted by Re z and b is called the *imaginary part* denoted by Im z *of the complex number z*. For example, if $z = 2 + i5$, then Re $z = 2$ and Im $z = 5$.

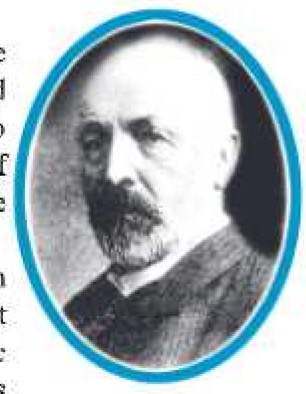
Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

Chapter 1

SETS

❖In these days of conflict between ancient and modern studies; there must surely be something to be said for a study which did not begin with Pythagoras and will not end with Einstein; but is the oldest and the youngest. — G.H. HARDY ❖

1.1 Introduction



Georg Cantor (1845-1918)

The concept of set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets.

The theory of sets was developed by German mathematician Georg Cantor (1845-1918). He first encountered sets while working on "problems on trigonometric series". In this Chapter, we discuss some basic definitions and operations involving sets.

1.2 Sets and their Representations

In everyday life, we often speak of collections of objects of a particular kind, such as, a pack of cards, a crowd of people, a cricket team, etc. In mathematics also, we come across collections, for example, of natural numbers, points, prime numbers, etc. More specially, we examine the following collections:

(i) Odd natural numbers less than 10, i.e., 1, 3, 5, 7, 9

(ii) The rivers of India

(iii) The vowels in the English alphabet, namely, *a, e, i, o, u*

(iv) Various kinds of triangles

(v) Prime factors of 210, namely, 2,3,5 and 7

(vi) The solution of the equation: $x^2 - 5x + 6 = 0$, viz, 2 and 3.

We note that each of the above example is a well-defined collection of objects in